

**Lesson
One****Modelling, Problem-
solving and Proof****Aims**

The aims of this lesson are to enable you to

- explore the principles of modelling and understand the problem-solving cycle
- know what constitutes proof and present it effectively
- understand what constitutes a necessary and sufficient condition
- distinguish between proofs based on deduction, contradiction and exhaustion
- consider what is required to disprove a conjecture

Context

The first lesson looks at mathematics itself, the assumptions we make, the skills we need to apply and the presentation of our work. This forms the basis of all that follows.



Most of these topics are covered in GCSE textbooks but also look at Bowles, *AQA A Level Maths: Year 1 / AS Student Book* (OUP, ISBN-13: 978-0198412953), Section 1.1.



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Modelling and the Problem-solving Cycle

These are general ideas that apply throughout the A-level syllabus. They are intended to show how Mathematics is applied to practical problems.

The problem-solving cycle tends to involve five stages:

Step 1: Problem specification

This involves expressing a problem in a mathematical form; for example, by setting up equations.

Step 2: Data collection

It may be necessary to carry out an experiment, in order to obtain data. The data may be used to determine a suitable model.

Step 3: Making assumptions

This may include simplifying assumptions (for example, that an object is a particle, or that gravity is constant).

Step 4: Obtaining results

This may involve solving equations.

Step 5: Interpretation

The practical application of the results is considered. If there is not sufficient agreement with experimental data, then assumptions may need to be amended, before repeating Step 4.

Proof

Necessary and sufficient conditions

Example 1

If p and q are even, then $p + q$ will be even.

This can also be written as:

p and q are even $\Rightarrow p + q$ is even,

where the symbol \Rightarrow stands for “implies that”.

We can also say that “ p and q being even is a sufficient condition for $p + q$ to be even”.

However, it is not a 'necessary' condition. Instead, both p and q could be odd.

Example 2

If a triangle is right-angled, then $a^2 + b^2 = c^2$ (where the shortest side is a and the longest side is c); i.e. Pythagoras' theorem.

The '**converse**' of Pythagoras' theorem is also true:

If $a^2 + b^2 = c^2$, then the triangle is right-angled.

If we denote "triangle is right-angled" by statement A , and $a^2 + b^2 = c^2$ by statement B , then

$A \Rightarrow B$ (or " A is a sufficient condition for B ")

and also $B \Rightarrow A$ or $A \Leftarrow B$ (" A is implied by B "; or " A is a necessary condition for B ").

For this example, we can write $A \Leftrightarrow B$ (" A implies and is implied by B ").

Alternatively, we can say that " A is a necessary and sufficient condition for B ". (Note however that $A \Rightarrow B$ is the 'sufficient' part, whilst $A \Leftarrow B$ is the 'necessary' part; i.e. it's the wrong way round! It might be less confusing to say " A is a sufficient and necessary condition for B ", but this is less commonly used.)

To complicate matters further, we can also say:

" A is true if and only if B is true" (often informally written as " A iff B ").

This is also the wrong way round: " A is true if B is true" is equivalent to $A \Leftarrow B$, whilst " A is true only if B is true" is equivalent to $A \Rightarrow B$.

Example 3

L = individual lives in London; E = individual lives in England

$L \Rightarrow E$ (but " $E \Rightarrow L$ " is not true)

L is a sufficient (but not necessary) condition for E

E is a necessary condition for L

Example 4

R = quadratic equation has repeated roots;

D = discriminant is zero

$$R \Leftrightarrow D$$

R is true if and only if D is true

R is a necessary and sufficient condition for D

Proof by deduction: Example

If $a > 1$ and $b > 1$, prove that $a + b < 1 + ab$

To prove this result, it isn't enough to show that it is true for a large number of cases.

A proof by deduction requires a logical argument. Here we could say, for example:

$$\begin{aligned} a > 1 \text{ and } b > 1 &\Rightarrow (a - 1)(b - 1) > 0 \\ &\Rightarrow ab - a - b + 1 > 0 \\ &\Rightarrow ab + 1 > a + b \\ &\Rightarrow a + b < 1 + ab \end{aligned}$$

Proof by contradiction

For the above example, we could say:

Suppose that $a + b \geq 1 + ab$.

Then $a - 1 \geq b(a - 1)$,

and hence $b \leq 1$ (since $a - 1 > 0$), which contradicts the initial assumption.

Proof by exhaustion

In some cases, it may be sufficient (and practicable) to examine all the possible situations that can arise.

Example

Prove that 53 is prime.

Here we need only show that 53 is not divisible by numbers up to 7.

Disproving conjectures

We may be asked to prove or disprove a **conjecture**.

Suppose that our conjecture is that $a + b \geq 1 + ab$ when $a > 1$ and $b > 1$.

Here we could consider the case where $a = b = 10$, and this is a **counter-example** that disproves the conjecture.

This isn't the only way of disproving a conjecture, but it is often the easiest.

Presentation of Proofs

Example

Prove that $\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{1}{\cos^2\theta\sin^2\theta}$

Avoid the following argument:

$$\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{1}{\cos^2\theta\sin^2\theta}$$

$$\Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} = \frac{1}{\cos^2\theta\sin^2\theta}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow 1 = 1$$

This isn't a watertight argument. We have shown that

$$\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{1}{\cos^2\theta\sin^2\theta} \Rightarrow 1 = 1,$$

but we really want to show that

$$1 = 1 \Rightarrow \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{1}{\cos^2\theta\sin^2\theta}$$

The situation could be remedied by replacing the \Rightarrow signs by \Leftrightarrow signs, but this style of proof is generally not thought to be very elegant.

In general, to prove that $A = Z$, we could show that

$$A = B = \dots = Z$$

But sometimes the best sequence isn't obvious, and it may be easier to prove the equivalent result that

$$A - Z = 0 \quad (\text{or sometimes } \frac{A}{Z} = 1).$$

This is especially true if A and Z are fractional expressions, as we then only need to show that the numerator of the resulting expression for $A - Z$ is zero.

Now try Activity One. This requires an understanding of trigonometric functions which you may find challenging if it is a while since your GCSE. Don't worry! Come back to it at a later point, if necessary.

Activity 1

What problems are there with the following arguments?



(i) Given that $0 \leq \theta < 360^\circ$,

$$\sin\theta = \tan\theta$$

$$\Rightarrow \sin\theta = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow 1 = \frac{1}{\cos\theta}$$

$$\Rightarrow \cos\theta = 1$$

$$\Rightarrow \theta = 0^\circ$$

(ii) $\frac{1}{x} < 2 \Rightarrow 1 < 2x$

(iii) $x - 6 = \sqrt{x}$
 $\Rightarrow (x - 6)^2 = x$
 $\Rightarrow x^2 - 13x + 36 = 0$
 $\Rightarrow (x - 9)(x - 4) = 0$
 $\Rightarrow x = 9 \text{ or } x = 4$

Activity 2

For each of the following, which of these statements is most appropriate: $A \Rightarrow B$, $A \Leftarrow B$, $A \Leftrightarrow B$, or none of these?



(i) $A: x = x^2$; $B: x = 1$

(ii) $A: x = 0 \text{ or } y = 0$; $B: xy = 0$

(iii) $A: x^2 > x$; $B: x > 1$

(iv) $A: b > a$; $B: b^2 > a^2$

Activity 3

Prove or disprove the following conjecture:



“The number 572 can be written in the form $n^3 - n$, for some positive integer n .”

Suggested Answers to Activities**Activity One**

- (i) It is not true that $\sin\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow 1 = \frac{1}{\cos\theta}$, as this assumes that $\sin\theta \neq 0$, which need not be the case.

Instead we can say:

$$\sin\theta = \tan\theta \Rightarrow \sin\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \sin\theta \left(1 - \frac{1}{\cos\theta}\right) = 0$$

$$\Rightarrow \text{either } \sin\theta = 0 \text{ or } 1 - \frac{1}{\cos\theta} = 0 \text{ (etc.)}$$

- (ii) It is only true if $x > 0$.
- (iii) $x = 4$ is a ‘spurious solution’; it doesn’t actually satisfy the original equation $x - 6 = \sqrt{x}$

$$\text{It is true that } x - 6 = \sqrt{x} \Rightarrow (x - 6)^2 = x,$$

but $x - 6 = -\sqrt{x} \Rightarrow (x - 6)^2 = x$ as well (and $x = 4$ is a solution of this equation).

Logically it is correct to say that the solution of $x - 6 = \sqrt{x}$ is either $x = 9$ or $x = 4$, as it is indeed one of the two.

Activity Two

- (i) $A \Leftarrow B$
- (ii) $A \Leftrightarrow B$
- (iii) $x^2 > x \Rightarrow x^2 - x > 0 \Rightarrow x(x - 1) > 0$
- \Rightarrow either $x > 0$ and $x - 1 > 0 \Rightarrow x > 1$
or $x < 0$ and $x - 1 < 0 \Rightarrow x < 0$;

i.e. $A \Rightarrow x > 1$ or $x < 0$,
so it isn't true that $A \Rightarrow B$;
And $x > 1 \Rightarrow x^2 > x$ (as $x > 0$, and hence both sides of the inequality
can be multiplied by x), so that $B \Rightarrow A$

Thus the answer is: $A \Leftarrow B$

- (iv) Considering the graph of $y = x^2$, we see that, if a and b are both positive, then $A \Rightarrow B$, but if a and b are both negative, then this isn't true. (If $a < 0$ and $b > 0$, for example, then it depends on the relative sizes of $|a|$ and $|b|$.)

Also, if $b^2 > a^2$ but $b < 0$ and $a > 0$, for example, then $B \Rightarrow A$ isn't true.

So none of the statements is true.

Activity Three

A proof by exhaustion could be considered, but the following may be quicker:

$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1) = (n - 1)n(n + 1)$, and one of the numbers $n - 1, n$ and $n + 1$ must be a multiple of 3, so that $n^3 - n$ is itself a multiple of 3. However, the sum of the digits making up 572 is not a multiple of 3, and so 572 itself is not a multiple of 3. Hence the conjecture is false.