Lesson 23 Trigonometry (1)

Aims

The aim of this lesson is to enable you to:

- work out the sine, cosine and tangent of acute angles
- use trigonometrical notation
- define the sine ratio
- use *cos* and *tan* as well as *sin*
- use the degree mode on your calculator

Context

Trigonometry introduces a new slant to our study of geometry and a variety of new ways to solve geometrical problems. Trigonometry deals with both the sides and angles of right-angled triangles.

Oxford Open Learning

Trigonometry

Introduction

This introduction is intended to provide some background for the topic:

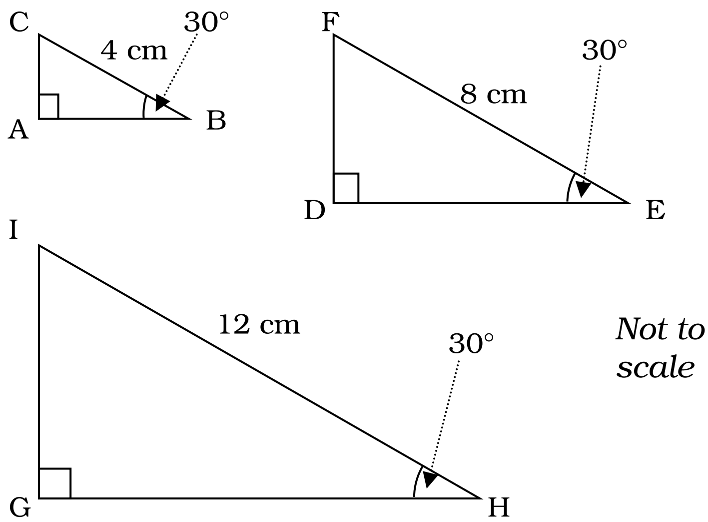
- what trigonometry is for
- an overview of how it works.

Trigonometry provides powerful methods for working with right-angled triangles. We already have one technique for working with right-angled triangles: Pythagoras' Theorem. However, Pythagoras' Theorem only deals with the **sides** of a right-angled triangle. Trigonometry deals with both the **sides and angles** of right-angled triangles.

The theory of Trigonometry depends on the notion of 'similarity'. Consider a right-angled triangle with an angle of 30°. In fact, this little bit of information is very important in the context of right-angled triangles. If one angle is 90° and another is 30°, then the third angle of the triangle must be 60° so that the three angles add up to 180°.

So the information that a right-angled triangle has an angle of 30° defines a family of triangles that all have the same three angles. All the triangles in this family have the same shape, and are therefore **similar** to each other.

The diagram shows three members of the family. They all have angles of 90°, 30° and 60°. The only aspect of the three triangles that differ is their size. The hypotenuses of the three triangles are 4 cm, 8 cm and 12 cm.



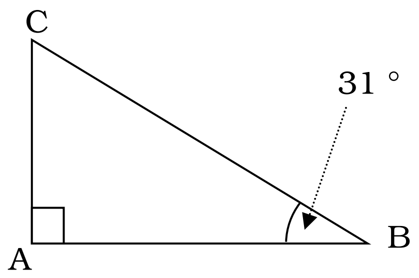
Now, for each triangle, consider the ratio of the side opposite the 30° angle to the hypotenuse. We would expect this ratio to be the same for all three triangles, since the triangles are similar. If you measure the triangles in the diagram, you should find that:

$$\frac{AC}{BC} = \frac{DF}{EF} = \frac{GI}{HI} = \frac{1}{2}$$

In other words, for any member of this family of right-angled triangles, the length of the side opposite the 30° is always half of the length of the hypotenuse.

What we have shown is that $\sin 30^0 = \frac{1}{2} = 0.5$, since 'sine' (abbreviated to sin) measures the ratio of the length of the side opposite an angle to the length of the hypotenuse.

Now 30° is a special angle with a convenient sine ratio of 0.5. Almost all other angles have a sine ratio that is a decimal that continues for ever without pattern or repetition. For instance if you draw **any** right-angled triangle ABC with an angle of 31° at B, then measure the lengths of the sides AC and BC and you should find that $\frac{AC}{BC} = 0.515$ correct to three decimal places.



In this lesson you will need lots of specific values of trigonometric ratios. You should be reassured that you do not need to draw a right-angled triangle with the correct angles and then measure the lengths of the sides on each occasion.

Once upon a time students used to look up trigonometric ratios, such as sine, in books of tables or else use a 'slide rule'. Nowadays, electronic calculators provide the 'sine' of an angle at the touch of a button. Curious students sometimes ask how the calculator knows all these values. The essential theory is covered in some of the higher level modules of A-level Mathematics. However, the precise implementation of the theory is probably a commercial secret. Even though the theory is well-known, calculating trigonometric ratios quickly and efficiently is a difficult problem.

Notation

When we do trigonometry, there is usually one angle in the right-angled triangle that we are working with. We either:

- know this angle
- or are trying to find this angle.

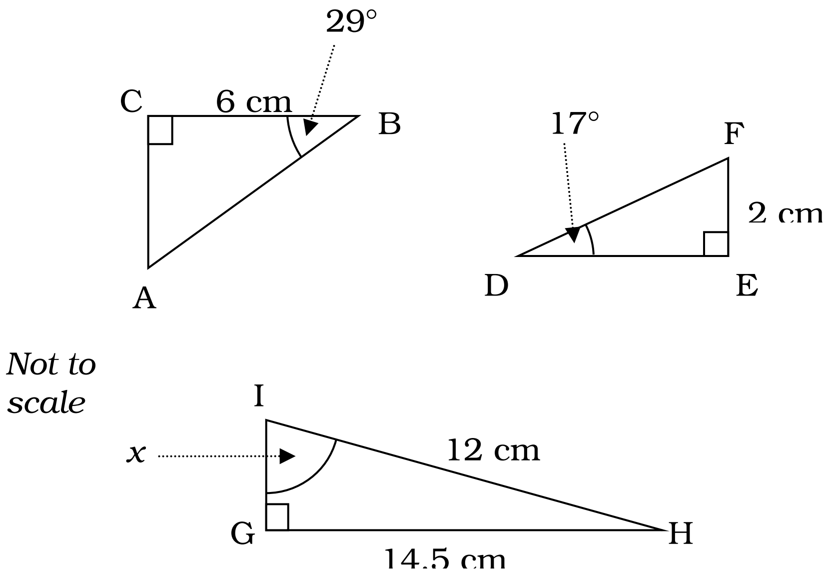
Log on to Twig and look at the film titled: **The Tunnel of Samos**

www.ool.co.uk/1704bu

How the ancient Greeks were able to use trigonometry to build the first aqueduct, tunnelling from two sides of a mountain to meet in the middle.

Example 1

Identify the angle we are 'working with' in each of the following triangles.



1. In triangle ABC we know angle ABC, so this is the angle we are 'working with'.
2. In triangle DEF we know angle EDF, so this is the angle we are 'working with'.
3. In triangle GHI, we want to know angle GIH, so this is the angle we are 'working with'.

The next stage is to label the three sides of the triangle, in the following order.

1. You already know the hypotenuse (H): the longest side which is opposite the right-angle.
2. The 'opposite' (O) side is opposite the angle we are 'working with'.
3. The 'adjacent' (A) is the side left over.

The 'adjacent' side is presumably so named because it is next to (hence adjacent) the angle we are working with. However, this is potentially confusing, since the hypotenuse is also next to the angle we are working with. There is no confusion, however, it you work in the order suggested: identify the hypotenuse first, then the opposite and finally the adjacent.

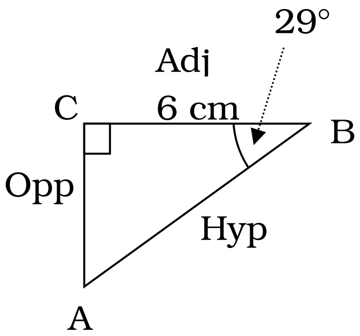
Example 2

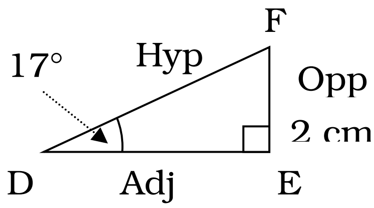
Label each of the sides of the three triangles in Example 1 with H, O and A.

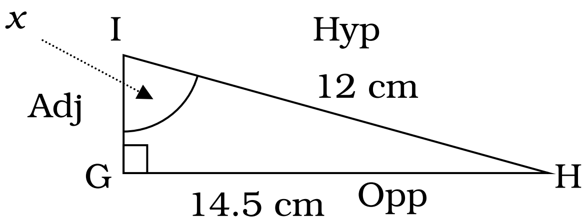
In triangle ABC, the hypotenuse is AB, opposite the right-angle. The side AC is 'opposite' the known angle of 29°. The third side, BC, must be the 'adjacent'.

In triangle DEF, the hypotenuse is DF, opposite the right-angle. The side EF is 'opposite' the known angle of 17°. The third side, DE, must be the 'adjacent'.

In triangle GHI, the hypotenuse is HI, opposite the right-angle. The side GH is 'opposite' the unknown angle *x*. The third side, GI, must be the 'adjacent'.







The Sine Ratio

We have already seen that the 'sine' of an angle is the ratio of the length of the side opposite the known angle to the length of the hypotenuse. However, it is easier to use the new notation to write:

$$\sin \text{ 'angle' } = \frac{\text{Opp}}{\text{Hyp}}$$

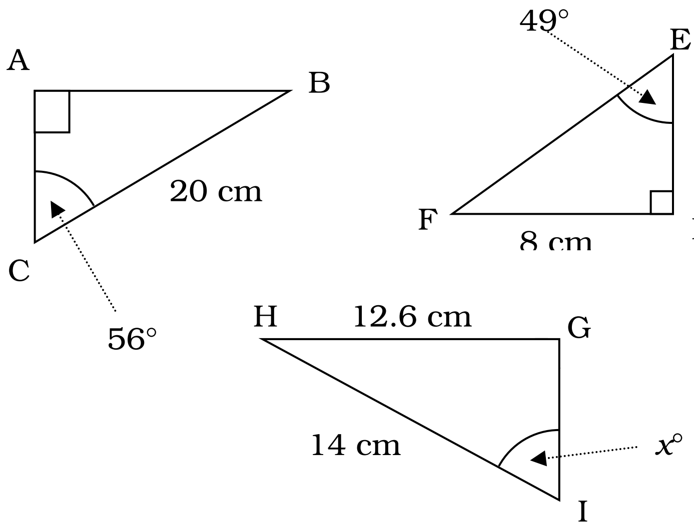
We will now use this formula to find missing sides and angles of right-angled triangles. The following routine will be used:

1. Identify the angle we are 'working with'

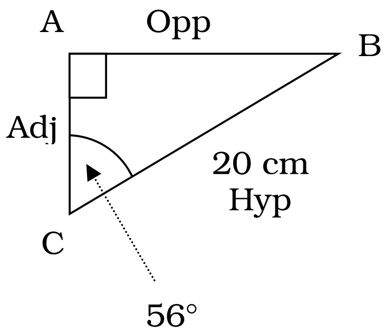
2. Label the three sides of the triangle with Hypotenuse, Opposite, Adjacent in that order
3. Substitute into the formula above any sides or angles that are known.
4. Use algebra and a calculator to find any missing side or angle.

Example 3

(a) Find the length of side AB in triangle ABC, correct to three significant figures.
(b) Find the length of side EF in triangle DEF, correct to three significant figures.
(c) Find the size of angle *x* in triangle GHI, correct to one decimal place.



(a) We are working with the known angle ACB, the 56°.
The hypotenuse is BC. The 'opposite' is AB. The 'adjacent' is AC.



We know the hypotenuse and the angle, so the formula becomes $\sin 56 = \frac{Opp}{20}$. We now do some simple algebra. We wish to isolate Opp, so move the 20 to the other side, changing the divide to a multiply:

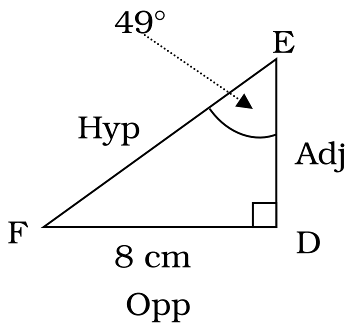
$$20 \times \sin 56 = Opp$$

Now use a calculator to obtain the answer, which should be AB = Opp = 16.6 cm, correct to three significant figures.

On older calculators, the routine is 20 [×] 56 [sin] .
On modern scientific calculators, the routine is different: 20 [×] [sin] 56 [=]. This version is much closer to what we write than the older version.

(b) We are working with the angle DEF = 49°.

The hypotenuse is EF. The side DF is 'opposite' the known angle 49°. The 'adjacent' side is DE.



Now write the formula, inserting the known values of Opp and the angle:

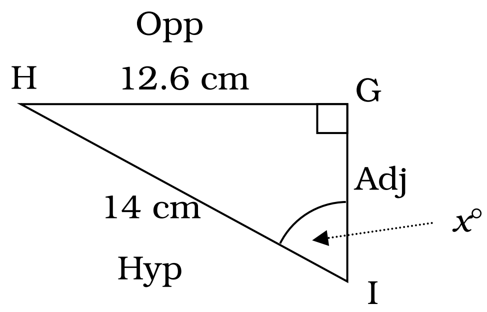
$$\sin 49 = \frac{8}{Hyp}$$

Now we need to do some algebra to isolate *Hyp*. If you look back to Lesson 12 you will find one special technique that helps in this situation, we can simply swap Hyp and sin 49. Perhaps it should be emphasised that sin 49 is one object, not two. It is **not** shorthand for sin × 49. It is just a single number (0.75470958… as it happens).

So $Hyp = \frac{8}{\sin 49}$. Using a calculator, we find that EF = *Hyp* = 10.6 cm, correct to three significant figures. (On a modern scientific calculator, the routine will be something like: 8 [÷] [sin] 49 [=] .)

(c) We are working with the unknown angle *x*, which we want to know.

The hypotenuse is HI. The 'opposite' side is GH. The 'adjacent' side is GI.



Write down the formula, inserting the known values of Opp and Adj:

$$\sin x = \frac{12.6}{14}$$

This example is convenient, since $\frac{12.6}{14} = 0.9$. The next stage is to write $x = \sin^{-1} 0.9$. However, this is less important than performing the final calculation. There are various ways, even on one type of calculator. One detail is essential, however. In order to find an angle, you need to identify the $\sin^{-1}$ function on your calculator. On most calculators this is a **combination** of two buttons. For instance, on a modern Casio, the combination is [SHIFT] [sin]. When you do this, the calculator display does indeed show "$\sin^{-1}$". On other makes of calculator, the combination can be [2nd Fn] [sin] or [INV] [sin] .

Various routines are possible on a modern Casio, for example:

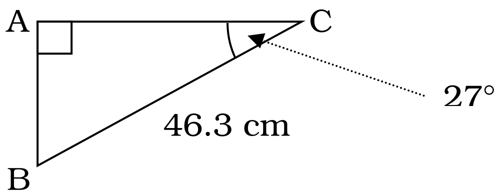
[SHIFT] [sin] 0.9 [=]
[SHIFT] [sin] [(] 12.6 [÷] 14 [)] [=]
[SHIFT] [sin] [(] 12.6 [$a\,b/c$] 14 [)] [=]
12.6 [÷] 14 [=] [SHIFT] [sin] [ANS] [=]

The first routine is not advised in general. This is because the number will normally be much less convenient than 0.9. There can then be problems deciding how many decimal places to write down and use without losing accuracy in the final answer. One way round this problem is to use the fourth routine, which uses the full accuracy available in the first calculation.

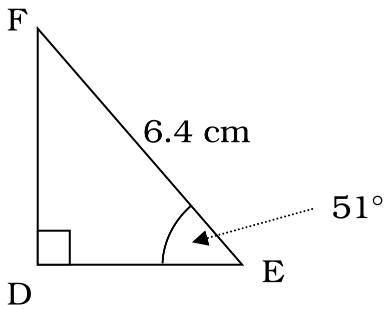
Note that the second and third routines are also very useful, since both calculations (division and $\sin^{-1}$) are performed by this single line. However, the brackets are **essential** in both these routines.

Activity 1

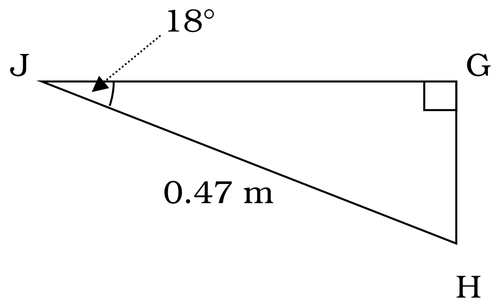
1 Find the length of AB correct to three significant figures.



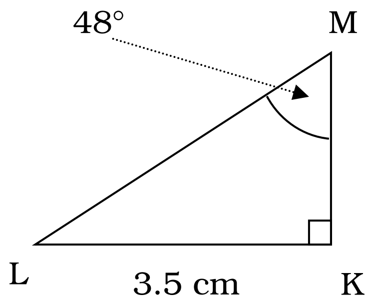
2 Find the length of DF correct to three significant figures.



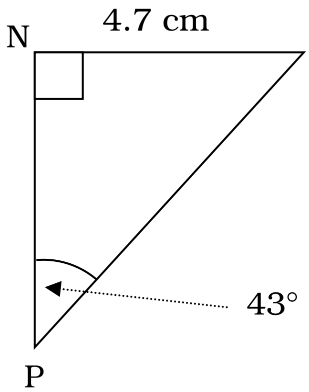
3 Find the length of GH correct to three significant figures.



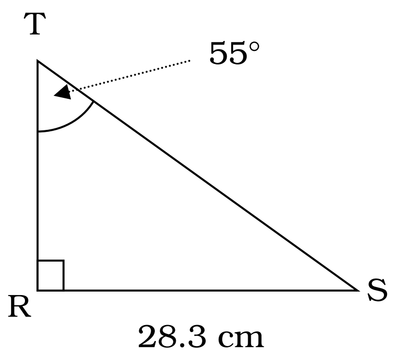
4 Find the length of LM correct to three significant figures.



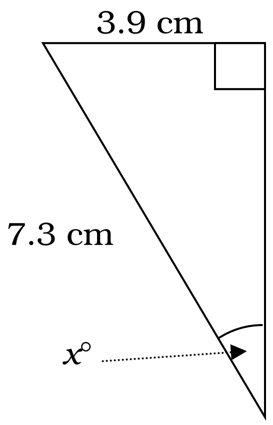
5 Find the length of PQ correct to three significant figures.



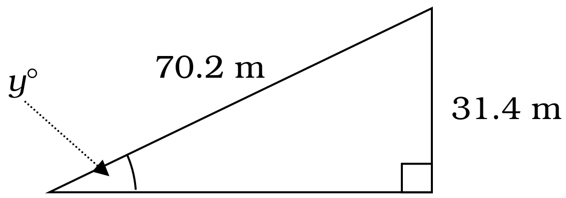
6 Find the length of ST correct to three significant figures.



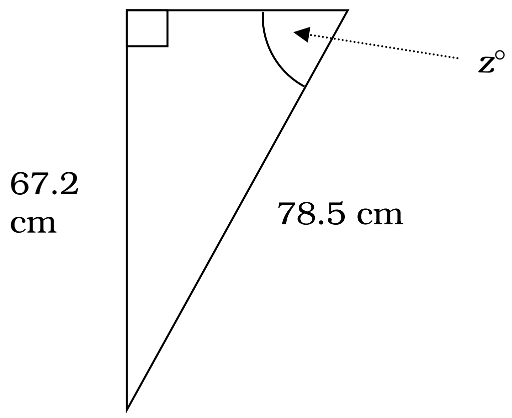
7 Find the size of the angle marked *x*, correct to three significant figures.



8 Find the size of the angle marked *y*, correct to three significant figures.



9 Find the size of the angle marked *z*, correct to three significant figures.



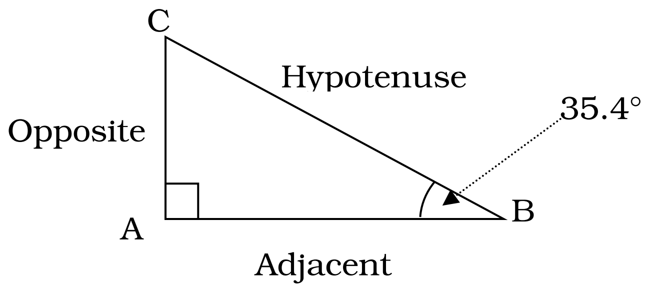
The Cosine and Tangent Ratios

Cosine (abbreviated to cos) is defined by $\cos = \dfrac{Adjacent}{Hypotenuse}$.

Tangent (abbreviated to tan) is defined by $\tan = \dfrac{Opposite}{Adjacent}$.

Let us just confirm what sin, cos and tan actually mean. If we draw a right-angled triangle ABC with the right-angle at A and, say, an angle of 35.4° at B, and then measure the three sides of the triangle, we should always find, regardless of the size of the triangle, that:

$\dfrac{AC}{BC} = 0.579$, $\dfrac{AB}{BC} = 0.815$ and $\dfrac{AC}{AB} = 0.711$ (correct to three decimal places). Why?



Firstly, as we already know, $\sin 35.4 = \frac{Opp}{Hyp} = \frac{AC}{BC}$. We also know how to find sin 35.4 on a calculator to give 0.579281172.....

Secondly, since $\cos = \frac{Adjacent}{Hypotenuse}$, we can write $\cos 35.4 = \frac{Adj}{Hyp} = \frac{AB}{BC}$. Using the cos button instead of the sin button, you should find that cos 35.4 = 0.815127795...

Finally, $\tan = \frac{Opposite}{Adjacent}$, so we can write $\tan 35.4 = \frac{Opp}{Adj} = \frac{AC}{AB}$. Using the tan button, you should find that tan 35.4 = 0.710663009...

Log on to Twig and look at the film titled: **Distance to the Sun and Moon**

www.ool.co.uk/1703eh

What are the trigonometric functions and how did the sine function allow Indian astronomers to prove that the Sun was 400 times farther from Earth than the Moon?

IGCSE Formula Sheet

Up until recently, the definitions of sin, cos and tan were on the IGCSE formula sheet. This is no longer the case! They now have to be remembered. However, this should not require a conscious effort. If you do enough practice with trigonometry questions, you should find that you have learned these formulae automatically.

Using cos and tan

The good news is that if you can use sin, then you can use cos and tan. There are no new techniques. The previous routine for finding missing sides and angles in right-angled triangles is **almost** enough. There is one new stage. This stage is simply deciding which of the three trig ratios (sin, cos or tan) to use.

The Decision: sin, cos or tan?

When we do trigonometry in right-angled triangles, we usually work with two sides only. However, there are two situations:

1. we know one side and want to find a second side
2. we know two sides (and want to find an angle).

It does not matter which of these two situations applies. The decision of sin, cos or tan is the same in both situations.

Working with the two sides:	**Trig ratio:**
Opp and Hyp	sin
Adj and Hyp	cos
Opp and Adj	tan

You do not need to learn this table, since the same information is available from the three formulae:

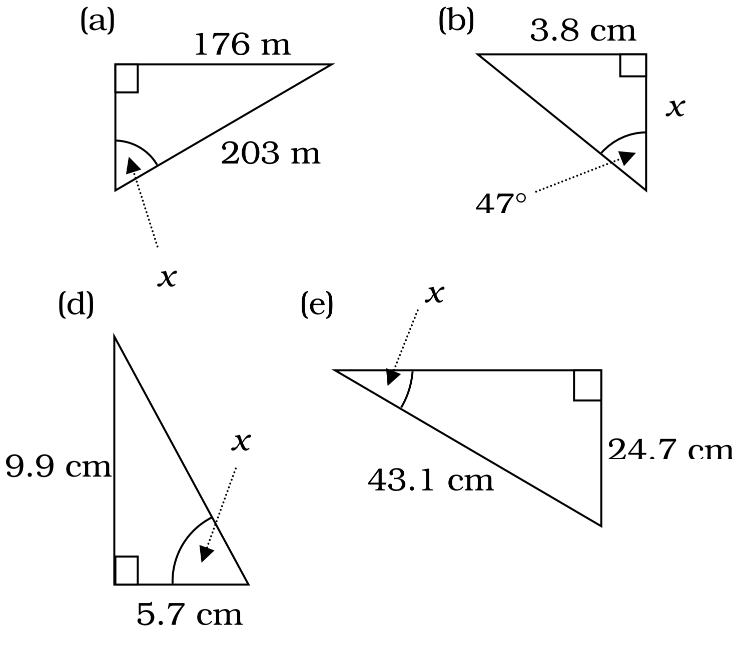
$$\sin = \frac{Opp}{Hyp} \qquad \cos = \frac{Adj}{Hyp} \qquad \tan = \frac{Opp}{Adj}.$$

All you need to do is look for the formula that contains the two sides you are working with.

First of all, the next Example concentrates on this new stage: the decision.

Example 1

Decide which of sin, cos or tan applies to each of the following triangles. Then stop, and do no calculations. Unknown sides and angles are marked *x*.



(a) We know opposite and hypotenuse. This is **sin**.
(b) We know opposite, we want to know adjacent. This is **tan**.
(c) We know adjacent, we want to know hypotenuse. This is **cos**.
(d) We know opposite and adjacent. This is **tan**.
(e) We know opposite and hypotenuse. This is **sin**.
(f) We know adjacent and hypotenuse. This is **cos**.

The Full Routine for Trigonometry

It is traditional to do four standard exercises on trigonometry. You have already done the first, using only sin. The next two are usually identical to the first, except that one is exclusively on cos and one on tan. A final exercise has all three jumbled up.

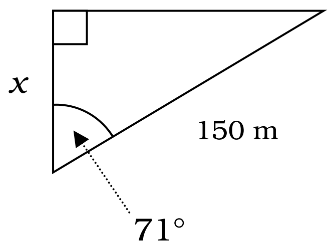
However, if you can use sin, then you can also use cos and tan, since the algebraic techniques required for all three are identical. The only difference is that you press different buttons on your calculator: cos or tan instead of sin. So we now proceed to Examples of the same type as the fourth of the traditional exercises. This uses the full routine for trigonometry, which is as follows. This is identical to the routine in the sin section, except for the addition of the 'decision' stage.

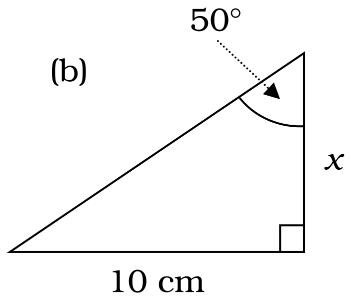
1. Identify the angle we are 'working with'.
2. Label the three sides of the triangle with Hypotenuse, Opposite, Adjacent in that order.
3. Decide which of sin, cos or tan to use.
4. Substitute into the appropriate formula any sides or angles that are known.
5. Use algebra and a calculator to find any missing side or angle.

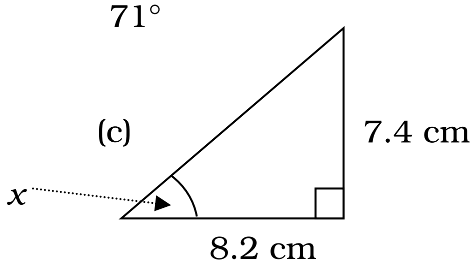
The aim of the above five-stage routine is to encourage you to think of trigonometry as **one theme with several variations** as opposed to lots of separate techniques.

Example 2

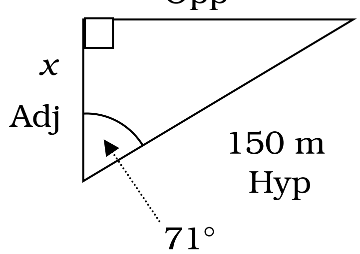
Find the length of the sides marked *x* and the size of the angle marked *x*.

(a)


(b)


(c)


(a) Identify the angle we are working with: this is the 71°.
Identify the hypotenuse (150 m), the side opposite the 71° and the adjacent (marked *x*).



We are working with x and 150m, i.e. adjacent and hypotenuse. This means cos.

Write down the formula $\cos 71 = \frac{\text{Adj}}{\text{Hyp}}$ inserting any known values:

$$\cos 71 = \frac{x}{150}$$

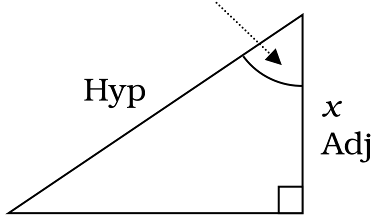
Do whatever algebra is necessary to isolate the unknown, x. We need to move the 150 to the left-hand side, changing the divide into a multiply:

$$150 \times \cos 71 = x$$

Using a calculator, we find that x = 48.8 m (correct to three significant figures). The calculator routine on a modern calculator is something like:

150 [×] [cos] 71 [=]

(b) Identify the angle we are working with: the 50°. Label the sides: the hypotenuse is opposite the right-angle, the 10 cm side is opposite the 50° so that the adjacent side must be the one marked x.



The two sides we are working with are the opposite and the adjacent. This means tan. Write down the formula $\tan 50 = \frac{\text{Opp}}{\text{Adj}}$ inserting the known values:

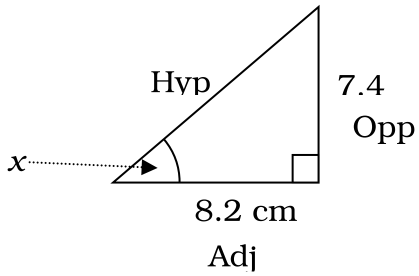
$$\tan 50 = \frac{10}{x}$$

We need to isolate x. The simplest way is to swap the x and the tan 50:

$$x = \frac{10}{\tan 50}$$

Using a calculator, we find that x is 8.39 cm, correct to three significant figures.

(c) Identify the angle we are working with: this is marked x. Label the sides: the hypotenuse is opposite the right-angle, the 7.4 cm side is opposite x and the adjacent is the 8.2 cm side.



The two sides we are working with are the opposite and adjacent. This means tan.

Write down the formula $\tan = \frac{Opp}{Adj}$ inserting the known values:

$$\tan x = \frac{7.4}{8.2}$$

The next line to be written should officially be:

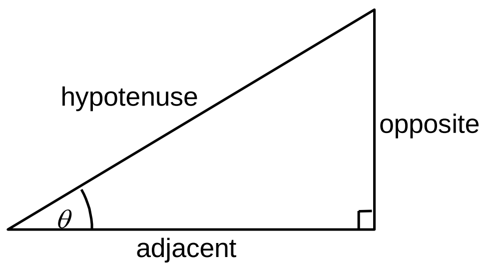
$$x = \tan^{-1}\left(\frac{7.4}{8.2}\right)$$

However, it is more important that you know how to use your calculator to obtain the final answer. One suitable calculator routine (for a modern Casio) is:

SHIFT tan (7.4 ÷ 8.2) =

You should find that the final answer is 42.1°, correct to three significant figures.

Here is a summarising formula sheet:



$$\text{adj} = \text{hyp} \times \cos\theta$$
$$\text{opp} = \text{hyp} \times \sin\theta$$
$$\text{opp} = \text{adj} \times \tan\theta$$

or

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

You are unlikely to be given this assistance in the examination. So it is handy to remember them, and you can use the mnemonic '**SOH, CAH, TOA**' to remind you (SOH: 'Sine equals Opposite over Hypotenuse' etc.).

Degree Mode

Finally, one small but essential aspect of calculator work. Most modern scientific calculators have three 'modes' for angles. You should see a small symbol on the calculator display to indicate the current mode.

Angle Mode	**Calculator Display**
Degrees	D
Radians	R
Gradians	G

It is essential that your calculator displays D while you are doing IGCSE Mathematics. It sometimes happens that a calculator seems to change mode by itself: this may be the result of a certain sequence of keys being pressed accidentally. In order to change mode on a modern Casio, press MODE twice. You are then given the option to select one of the three modes in the above table. Obviously, you would choose the option for 'Deg'. Make sure that you know how to change the mode on your own calculator to degrees.

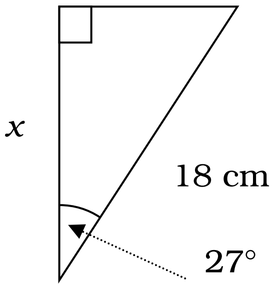
This last paragraph is for background information only. The 'radian' is a different unit of measurement for angles. Radians are important in A-level Mathematics. There are 360 degrees in a full circle: the reason is an historical accident – the ancient Babylonians counted in units, 60s and 360s instead of our familiar units, tens and hundreds. There are 2π (in other words, approximately 6.283185307…) radians in a full circle.

This might seem at first to be inconvenient. However, the radian is a 'natural' measure for angles in higher mathematics. Indeed, if we ever meet or otherwise make contact with extraterrestrials who are at least as technologically advanced as ourselves, then it is almost certain that we would find that they measure their angles in radians. There are 100 gradians in a right angle, instead of the usual 90 degrees. This was an unsuccessful attempt to make angle measurement 'metric', although some German engineers still use gradians.

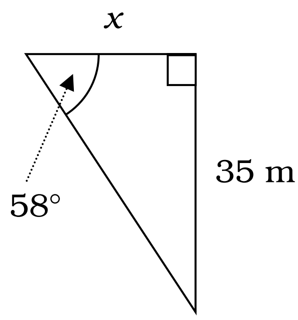
Activity 2

1 Decision practice. No calculation required. Each triangle has a side or angle marked *x*. Just write down which of sin, cos or tan you need to use **if** you needed to find the value of the unknown *x*.

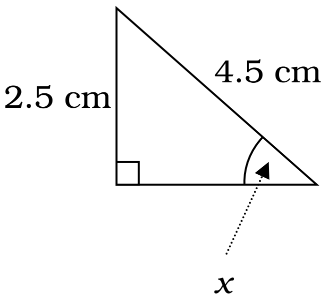
(a)


x
18 cm
27°

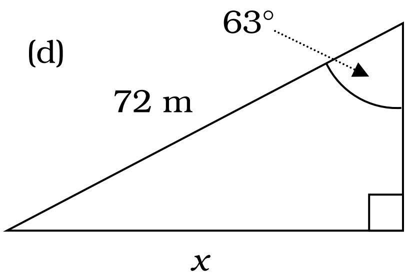
(b)


x
35 m
58°

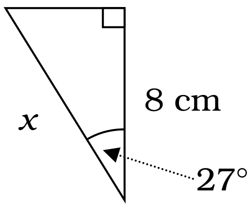
(c)


2.5 cm
4.5 cm
x

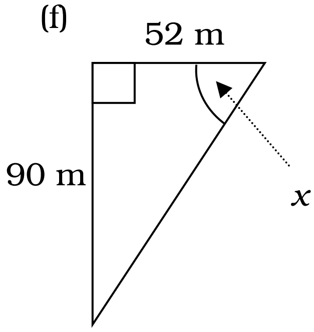
(d)


63°
72 m
x

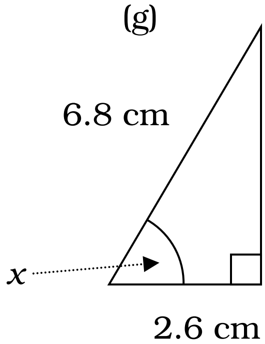
(e)


x
8 cm
27°

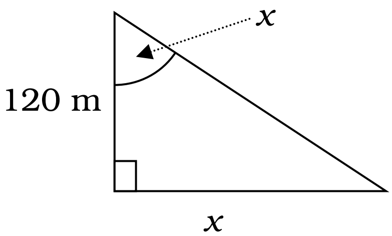
(f)


52 m
90 m
x

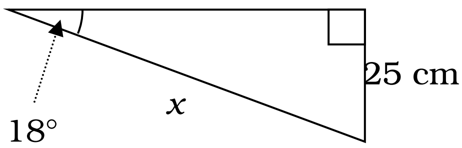
(g)


6.8 cm
x
2.6 cm

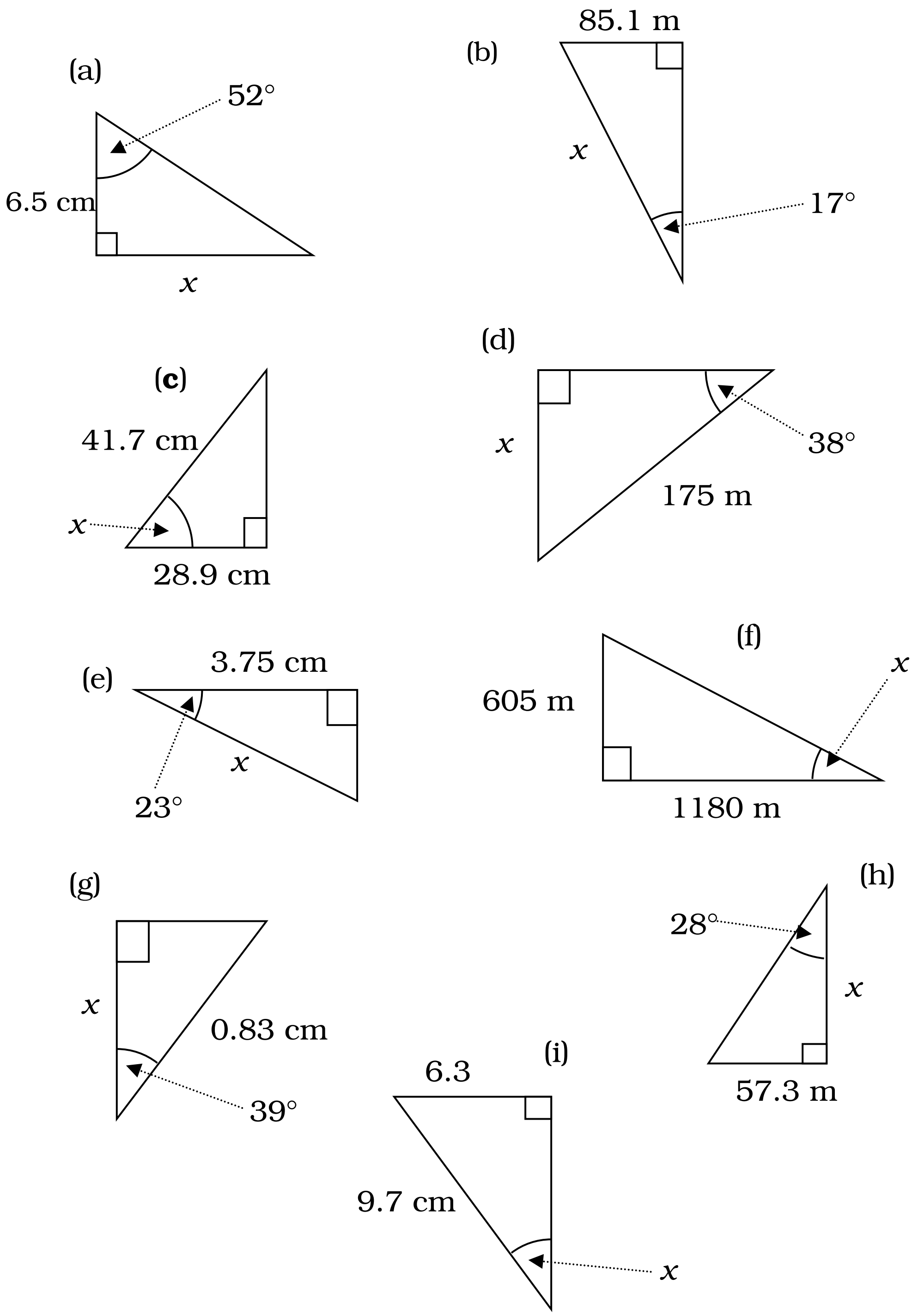
(h)


x
120 m
x

(i)


25 cm
x
18°

2 Find the length of all sides marked x, correct to three significant figures. Find the size of all angles marked x correct to one decimal place.



Suggested Answers to Activities

Activity One

1 21.0 cm
2 4.97 cm
3 0.145 m
4 4.71 cm
5 6.89 cm
6 34.5 cm
7 32.3°
8 26.6°
9 58.9°

Activity Two

1. (a) cos
   (b) tan
   (c) sin
   (d) sin
   (e) cos
   (f) tan
   (g) cos
   (h) tan
   (i) sin

2. (a) 8.32 cm
   (b) 291 m
   (c) 46.1°
   (d) 108 m
   (e) 4.07 cm
   (f) 27.1°
   (g) 0.645 cm
   (h) 108 m
   (i) 40.5°