Lesson Thirty Four

Getting to Grips with Graphs (1)

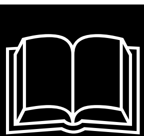
Aims

The aims of this lesson are to help you to:

- understand the purpose of some more complex forms of graph
- identify the gradient of linear graphs
- calculate the gradient of a sloping straight line
- produce graphs to help solve linear equations
- use graphs to solve quadratic equations

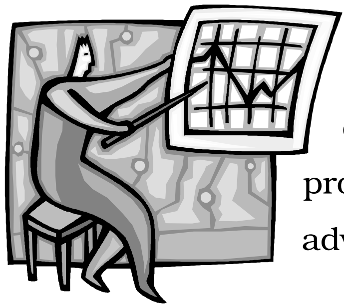
Why am I studying this?

Having looked at the common types of chart, we now move specifically to Graphs. These can be a useful 'bridge' to help us picture, and analyse, the results of a more complex equation or sequence of data. In this and the following lesson we shall discover how such ideas connect: how to identify and describe the types and features of line graphs, *eg* their 'gradient' and 'intercept', and how to predict and plot our own, based on given data or formulae.



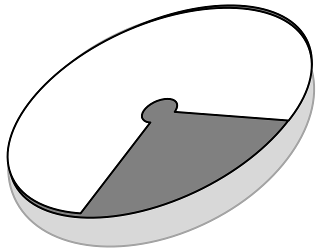
The topics in this lesson are covered in *Key Stage 3 Maths Complete Revision and Practice* on pages 71-93.

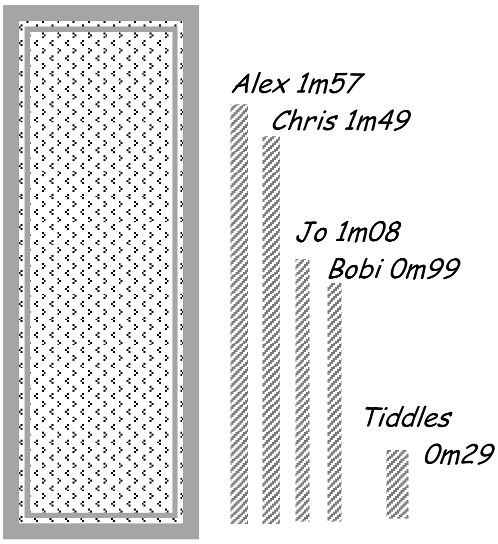
Graphs



While **pie-charts** and **bar-charts** are very useful in everyday life as a means of presenting information, they are probably not as important in more advanced mathematics as ***graphs***.

A simple pie-chart to show that ¾ of people haven't yet helped themselves to pudding ...

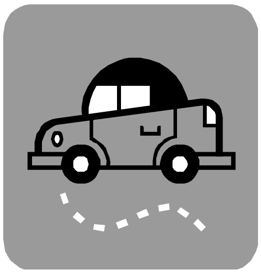




A simple bar-chart from the bathroom door-post showing how tall we are at home

Working with graphs requires a much more theoretical (some would say, 'mathematical') approach, and it is vital that you gain as much practice as possible at drawing graphs. Make sure you use graph paper with 2 mm divisions, like the type used in examinations. In the first section on this subject we are going to concentrate on drawing graphs from given data.

Bar charts are likely to be used when the data is **discrete**, in other words when only **specific values** can be used.



Let's take, for example, a graph of the number of passengers in cars. It would not be sensible to talk about 3.76 passengers; and in such cases we cannot sensibly join the points up to each other.

However, if the data is **continuous** then it often makes sense to link the points in some kind of line. For example, temperature graphs could be joined up in this way – because temperatures can have any value over a given range, and they vary over time (*eg* with the weather, or the body temperature of someone who's unwell). It would be quite logical to plot this on a graph to show rising or falling **trends**, in a form that is simple to grasp, and in this context it might be fully reasonable to talk about a temperature of 31.75 °C, which could be tricky to show accurately on a block chart.

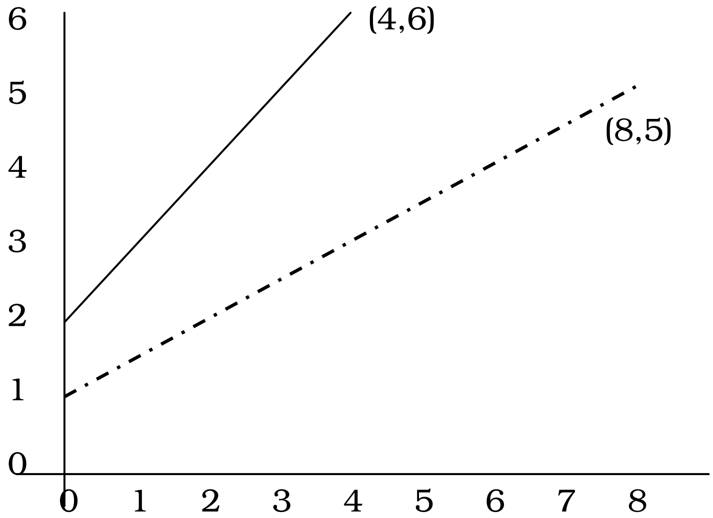
Graphs of Linear Equations

The word 'linear' means 'straight-line-forming'. What sort of equation is linear? $y = 3x + 5$ would be linear, or $y = 2x - 3$, but not $y = x^2 + 2x + 3$ or any other with powers of x (or y) in it.

As the name suggests, linear equations like $y = 3x + 5$ are represented graphically by a straight line whereas $y = x^2 + 2x + 3$ would be shown by a curve.

If we picked a relatively simple linear equation such as $y = x + 2$, you would have had little trouble working out the points, because if $x = 1$ then $y = 1 + 2 = 3$, and if $x = 2$ then $y = 4$ and so on. You would plot a line running between the points (1,3) and (2,4) and beyond. It would, for instance, presumably cut back through the y-axis at (0,2) because this point still has a y-value 2 greater than its *x-value*.

This line is shown 'solid' in the plot below, up to the value of (4,6):



The other line shows the equation $y = (0.5x + 1)$, which may sound a bit more complex, but it is still linear:

If $x = 1$, we halve that to give 0.5 and then add 1, making 1.5
If $x = 2$, we halve that to give 1 and add another 1, making 2
If $x = 3$, we halve that to give 1.5 and then add 1, making 2.5

We may present this more succinctly in table form, with the working stages shown:

x	1	2	3	4
$0.5x$	0.5	1	1.5	2
$0.5x + 1$ (= y)	1.5	2	2.5	3
Coordinates	(1, 1.5)	(2, 2)	(3, 2.5)	(4,3)

Clearly enough the line is rising by steps of ½ in a straight line: you could work out as many points as you like and plot them, they would all fall on that same line (even a 'clever' point like $x = 2.8$, in which case $y = 2.4$; try it if you must!).

You would not have major problems identifying a linear equation (*ie* with no 'powers' on any of its terms), and working out *at least three* points in a little table such as the one just above.

The reason for checking 'at least three' is to make sure they indicate a straight line, and plot it out as one. If there's any suggestion of a kink, you should re-check your calculations!

Activity 1

Make out tables, and plot graphs, for the following equations.

If you spot the one non-linear 'decoy' equation in this Activity, please *don't* (at this stage) go ahead with planning or plotting It; It's just to keep you on your toes. We'll deal with that kind a little later in the lesson!

1. $y = x + 3$
2. $y = 1.5\,x$
3. $y = x^2 - 1$
4. $y = 6 - x$
5. $2x = y + 1$

Using Graphs to Solve Equations

Linear equations are all very well – and we hope you had little trouble combining algebra with the plotting, to produce the straight lines in Activity One, where the range of values was also kept fairly simple.

You could probably describe the line they each make, either in mathematical or 'plain' terms, without too much difficulty; 'up two for every one, starting from three', or whatever.

Some equations, particularly any that are non-linear, can be harder to solve by algebraic techniques. In this case, we can definitely use graphical methods, plotting some points and then joining them up, usually with a smooth curve.

Back in our early work with shapes and area, you may recall that once we let go of straight lines, matters became a bit more complex: a bit like the moment when you let go of the firm, straight side of a swimming-pool or ice-rink, and try some freer movements on your own – along, perhaps, with a slight temptation to panic. There's nothing particularly frightening about graphs with curves in them; they're just different, with different origins and purposes, and once again (dare we say) you may find them surprisingly elegant once you become used to them. You will recognise certain familiar tell-tale shapes and get comfortable with 'joining the dots' to form them.

The following example is just such a **quadratic** equation, i.e. one that contains a 'square' term. The word 'quadratic' may helpfully put you in mind of quadrilaterals – perhaps with right-angled corners, and certainly some 'square area' value as opposed to the one-dimensional line of a linear equation.

Example 1

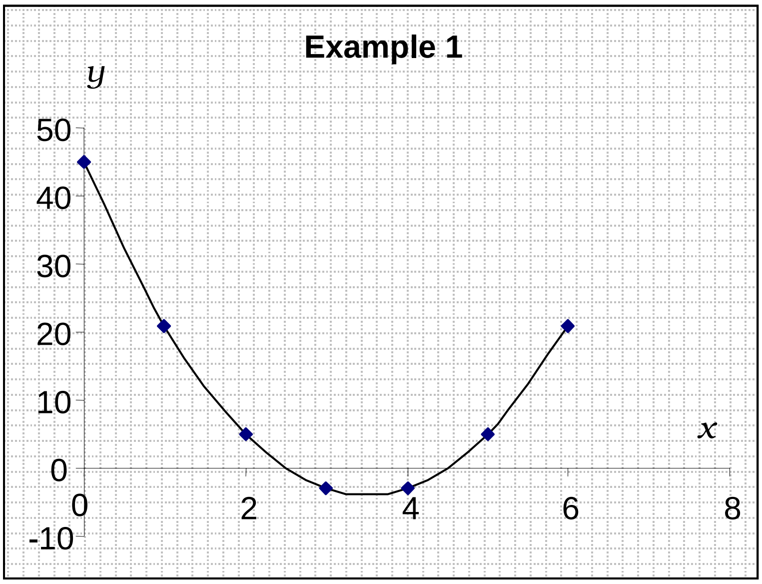
The following table of values is for the function $y = 4x^2 - 28x + 45$.

X	0	1	2	3	4	5	6
$y = 4x^2 - 28x + 45$	45	21	5	-3	-3	5	21

(a) Plot the points on a graph, and join up to make a smooth curve.

(b) Solve the equation $4x^2 - 28x + 45 = 0$.

(a)



(b) To solve the equation $4x^2 - 28x + 45 = 0$, we are looking to see where $y = 0$ on the graph. In other words, where does the curve cross the x axis? It looks as though the curve crosses the x axis when x is either 2½ or 4½.

Example 2

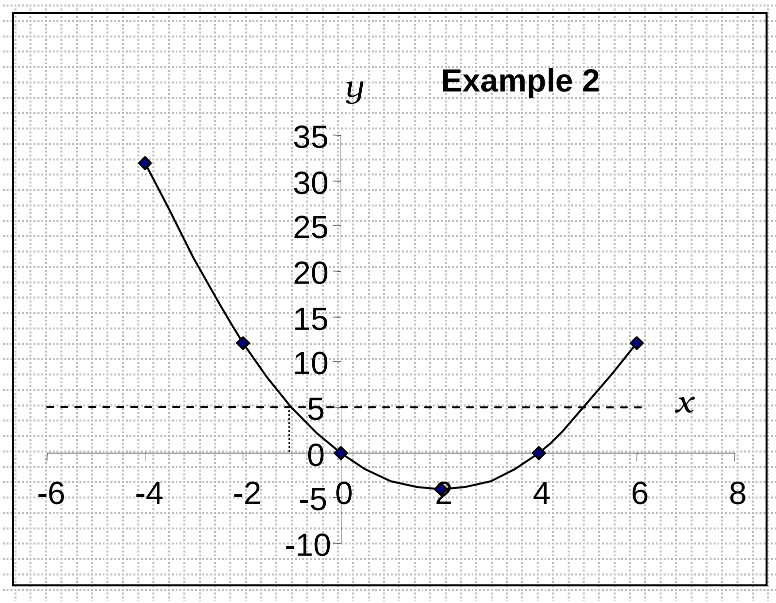
The following table of values is for the function $y = x^2 - 4x$.

x	-4	-2	0	2	4	6
$y = x^2 - 4x$	32	12	0	-4	0	12

(a) Plot the points on a graph, and join up to make a smooth curve.

(b) Solve the equation $x^2 - 4x = 5$.

(a)



(b) We need to solve $x^2 - 4x = 5$. This is the same as finding the points on the curve that have a y value of 5. In other words, where does the curve cross the horizontal line $y = 5$? We should see that this happens when x is -1 or 5.

If this appears a bit complex, and perhaps daunting, to you, try and cling onto the following features which you may have noticed (by all means go back and check them):

- If the first term is an x^2, there will probably be a 'saggy' curve as in the two examples we've seen. Its technical name is a **parabola** and it may put you in mind of similar real-life shapes, such as the one made by dangling ropes.
- If you look into the relatively plain series for $y = x^2$, pure and simple and without any other complications, you can satisfy yourself that its graph would soar away through (1,1), (2,4), (3,9), (4,16), (5,25) and onwards and upwards at an ever-increasing rate (by way of (10, 100) for instance).
- This same series would run 'backwards and upwards' into the negative zone, because $(-n)^2 = -n \times -n$, and any two negative

numbers multiplied together will give a positive product (*e.g.* –4 × –4 = 16, not ' –16 '). Hence the parabola is symmetrical, reflecting in the vertical line that runs straight upwards from its 'tip' point: it goes 'back and up' to match its 'forwards and up'.

- The parabolas that we've seen seem to suggest, visually, something that comes swooping down and then heads back up again – a bit like the flight of a seabird catching fish, perhaps; or you might prefer to think of it as the tip of the tongue of a pet as it laps from its water-bowl. Whichever way you look at it, these curves either touch down onto a flat line, or dip beneath it and come back up again to break that line a second time.
- There may well be real-life reasons for us wanting to discover at what point a curve touches a certain level, dips below it or passes back up through it. A 'controlled dip' in some financial or industrial process, for instance, might need careful judgment and planning to see that levels of money, temperature or components did not run down into a danger zone – or for how long such a crisis might last before they were in the clear again.
- Careful working of the algebra should still, always, enable you to make a table of points from which you can mark and draw the curve. At this stage we will take care of those calculations for you: all you need to do is to plot the dots and link them up.
- You may feel more comfortable, once you've drawn the points, if you then turn your paper away from you by 180°. Your hand will then be on the *inside* of the parabola, which should make it easier to join up in one smooth action.

Activity 2

1 The following table of values is for the function $y = x^2 - 2x - 15$.

x	-4	-2	0	2	4	6
$y = x^2 - 2x - 15$	9	-7	-15	-15	-7	9

(a) Plot the points on a graph, and join up to make a smooth curve.
(b) Solve the equation $x^2 - 2x - 15 = 0$.

2 The following table of values is for the function $y = 13x - 2x^2$.

x	0	2	4	6	8
$y = 13x - 2x^2$	0	18	20	6	-24

(a) Plot the points on a graph, and join up to make a smooth curve.
(b) Solve the equation $13x - 2x^2 = 15$.

3 (a) Copy and complete the following table of values which is for the function $y = x^2 - 11x + 26.25$.

X	1	2	3	4	5	6	7	8	9
$y = x^2 - 11x + 26.25$	16.25	8.25	2.25						

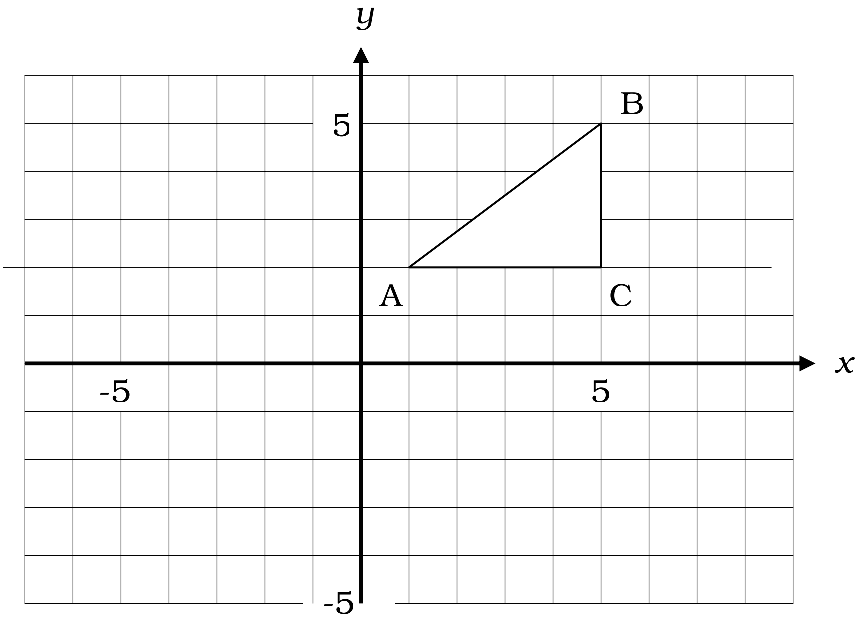
(b) Plot the points on a graph, and join up to make a smooth curve.
(c) Solve the equation $x^2 - 11x + 26.25 = 0$.

Co-ordinates

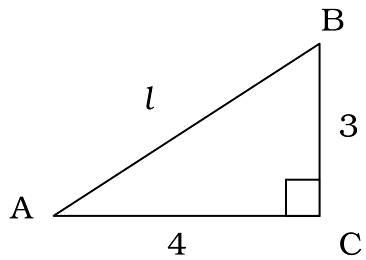
Meanwhile, if we need to find the length of a slanting line segment AB, we can use Pythagoras' Theorem.

Example 3

A and B have co-ordinates (1, 2) and (5, 5) respectively. Find the length of AB.



The points A and B are plotted on the grid. The point C (5, 2) is also plotted. The triangle ABC is right-angled. We know two sides, and can find the third side AB using Pythagoras' Theorem.



We see from the diagram that AC is 4 units and that BC is 3 units. In fact, the length of AC is the **difference** between the x co-ordinates of A and B, and the length of BC is the **difference** between the y co-ordinates of A and B.

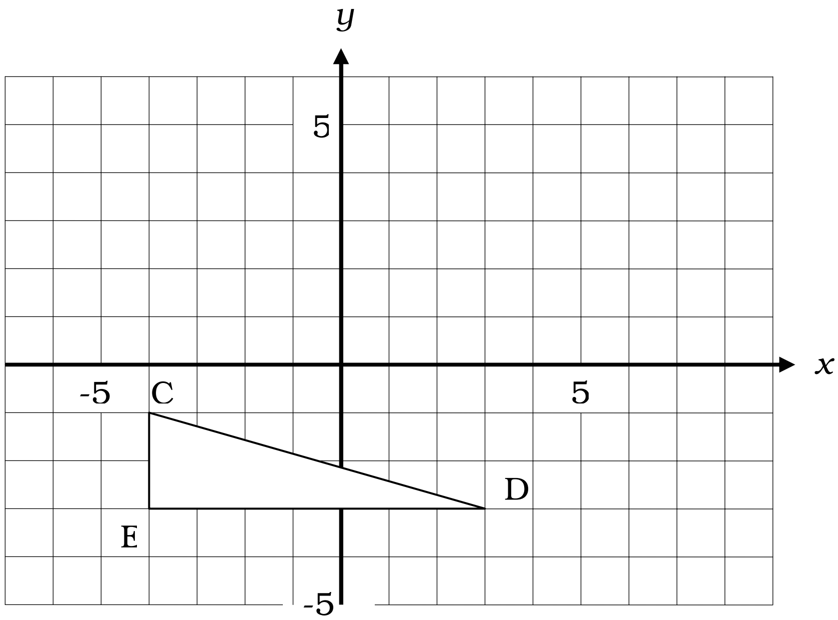
Let AB be l. AB is the hypotenuse of the right-angled triangle. Using Pythagoras' Theorem:

$$l^2 = 4^2 + 3^2$$
$$l^2 = 16 + 9 = 25$$
$$l = \sqrt{25} = 5$$

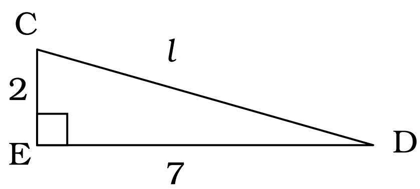
So the length of the line segment AB is 5 units.

Example 4

C and D have co-ordinates (-4, -1) and (3, -3) respectively. Find the length of CD, correct to three significant figures.



Plot the points C and D on a grid. Also plot the point E, so that CDE is a right-angled triangle.



From the grid we see that the 'difference between the x co-ordinates' of C and D is 7: this is the length of DE. Similarly, the 'difference between the y co-ordinates' of C and D is 2: this is the length of CE.

Let the length of CD be l. Then by Pythagoras' Theorem:

$$l^2 = 7^2 + 2^2$$
$$l^2 = 49 + 4 = 53$$
$$l = \sqrt{53}$$

Using a calculator, the length of CD is 7.28 units, correct to three significant figures.

Mid-point of AB

If we know the co-ordinates of A and B, we can easily find the co-ordinates of the mid-point of AB. The rule is to find the **average** x co-ordinate of A and B and the **average** y co-ordinate of A and B. (An average, to put it simply, is where you take a number of unequal quantities, add them all together and divide by how many items there were in the first place. We met the idea of 'averaging' just two numbers, ages ago, while we were calculating the so-called waistline of a trapezium – by adding its long & short parallel sides and then dividing by 2.)

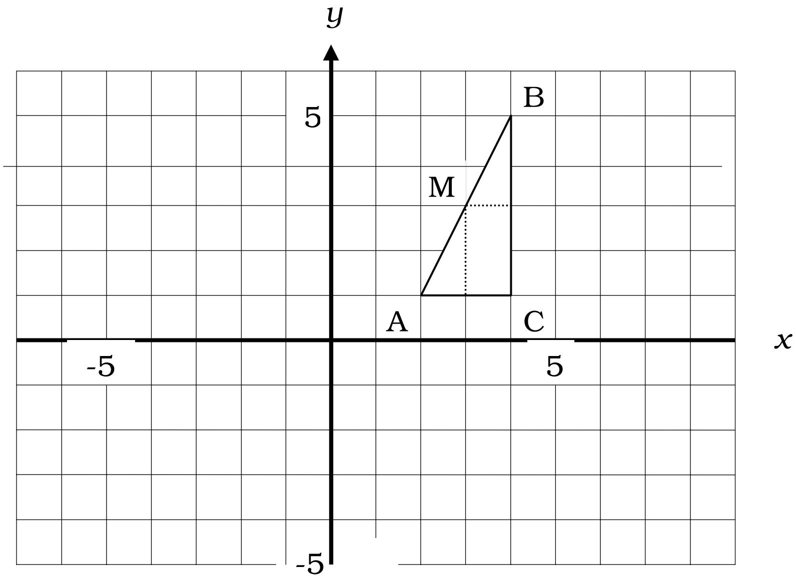
Example 5

A and B are the points with co-ordinates (2, 1) and (4, 5) respectively. Find the co-ordinates of the mid-point of AB.

The rule says that the mid-point of AB is $\left(\frac{2+4}{2}, \frac{1+5}{2}\right)$ which simplifies to (3, 3).

Let us look at the use of the rule in more detail. If M is the mid-point of AB, then the x co-ordinate of M is the average of the x co-ordinates of A and B: $\frac{2+4}{2} = 3$. Similarly, the y co-ordinate of M is the average of the y co-ordinates of A and B: $\frac{1+5}{2} = 3$.

The diagram shows that the rule has worked: the mid-point of AB is clearly (3, 3).



The rule also works when the co-ordinates are negative.

Example 6

Find the co-ordinates of the mid-point of AB when A and B have co-ordinates:

(a) (–2, 7) and (4, 11) (b) (3, –1) and (–9, –13)

(a) The average of the x co-ordinates is $\frac{-2+4}{2}=\frac{2}{2}=1$. The average of the y co-ordinates is $\frac{7+11}{2}=\frac{18}{2}=9$. The mid-point therefore has co-ordinates (1, 9).

(b) The average of the x co-ordinates is $\frac{3+(-9)}{2}=\frac{-6}{2}=-3$. The average of the y co-ordinates is $\frac{-1+(-13)}{2}=\frac{-14}{2}=-7$. The mid-point therefore has co-ordinates (–3, –7).

Activity 3

1 *Without using a calculator,* find the length of the line AB in each of the following cases:

(a) A(2, 9), B(11, 21)

(b) A(4, 7), B(–4, 1)

(c) A(–2, 6) B(1, 2)

(d) A(–4, –8), B(1, 4)

(e) A(–6, –4), B(–2, –1)

2 Use a calculator to find the length of the line AB in each of the following cases, correct to three significant figures:

(a) A(9, 13), B(3, 4)

(b) A(2, 5), B(8, –1)

(c) A(–4, 8), B(3, –2)

(d) A(–2, –6), B(–1, –9)

3 *Without using a calculator,* find the co-ordinates of the mid-point of the line AB in each of the following cases:

(a) A(4, 7), B(10, 11)

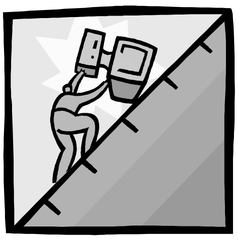
(b) A(3, –4), B(–1, 2)

(c) A(–4, –1), B(–2, –5)

(d) A(4.8, 1.3), B(5.4, 0.7)

(e) A(1½, 3¼), B(3, 2¼)

Gradient

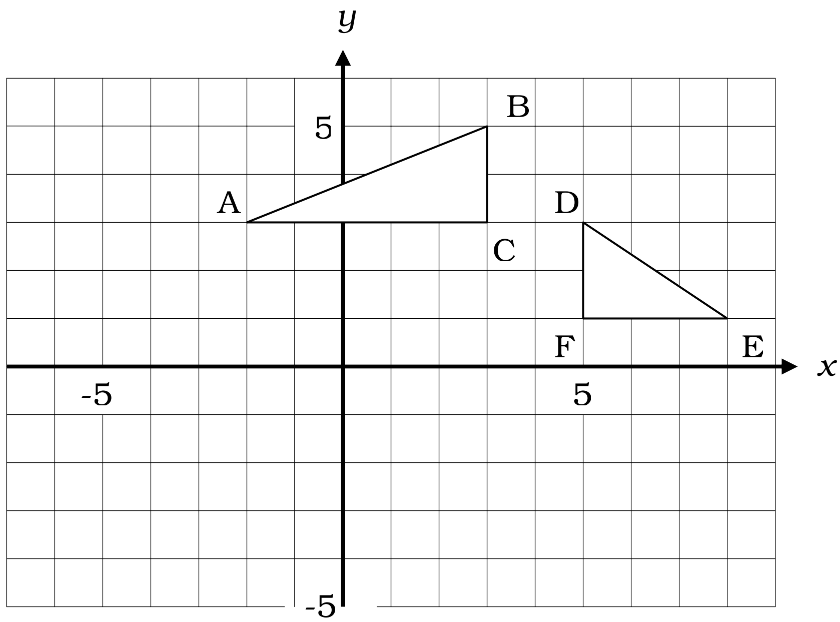


Gradient is a measure of slope: how steep a line is. The gradient of a line segment AB is defined as $\frac{\text{increase in } y}{\text{increase in } x}$ as we travel from A to B. (It turns out that it makes no difference if we travel from B to A: the result is the same.)

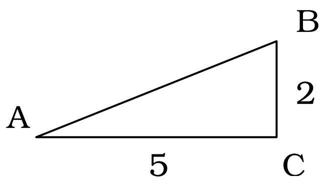
Example 7

(a) Find the gradient of the line segment AB where A and B have co-ordinates –2, 3) and (3, 5) respectively.

(b) Find the gradient of the line segment DE where D and E have co-ordinates (5, 3) and (8, 1) respectively.

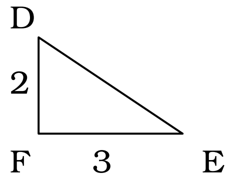


(a) The points A and B area plotted on the above grid. The point C is also plotted, so that ABC is a right-angled triangle.



We see from the grid that the lengths of AC and BC are 5 and 2 units respectively. AC represents the increase in x, and BC represents the increase in y as we travel from A to B. The gradient of AB is therefore $\frac{2}{5}$ or 0.4.

(b) the points D and E have been plotted on the grid together with the point F(5, 1), so that DEF is a right-angled triangle.



We see from the grid that the lengths of EF and DF are 3 and 2 units respectively. The increase in x is certainly 3. However, as we travel from D to E, y does not increase: it decreases. We therefore say that the 'increase in y' is negative. The gradient of DE is $\frac{-2}{3}$ or $-\frac{2}{3}$.

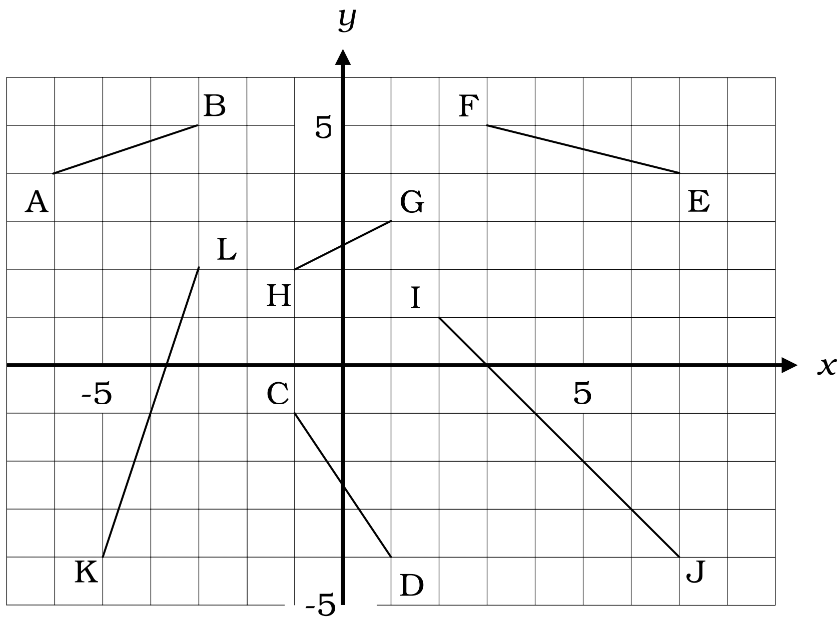
Positive or Negative Gradient?

Positive gradient is 'uphill' and negative gradient is 'downhill'. However, we need to be clear about the direction of travel when deciding the sign of the gradient. We must travel so that x increases. So for the points A and B, we travel from A to B, and see that we are going uphill, and the gradient is *positive.* For the points D and E, x increases as we travel from D to E. We see that we are going downhill from D to E, hence the **negative** gradient. In practice, all we do is to move left to right, and ask whether we are going uphill or downhill.

Example 8

Which of the following line segments have:

(a) a positive gradient (b) a negative gradient?



(a) The line segments with a positive gradient are: AB, GH and KL.

(b) The line segments with a negative gradient are: CD, EF and IJ.

Activity 4

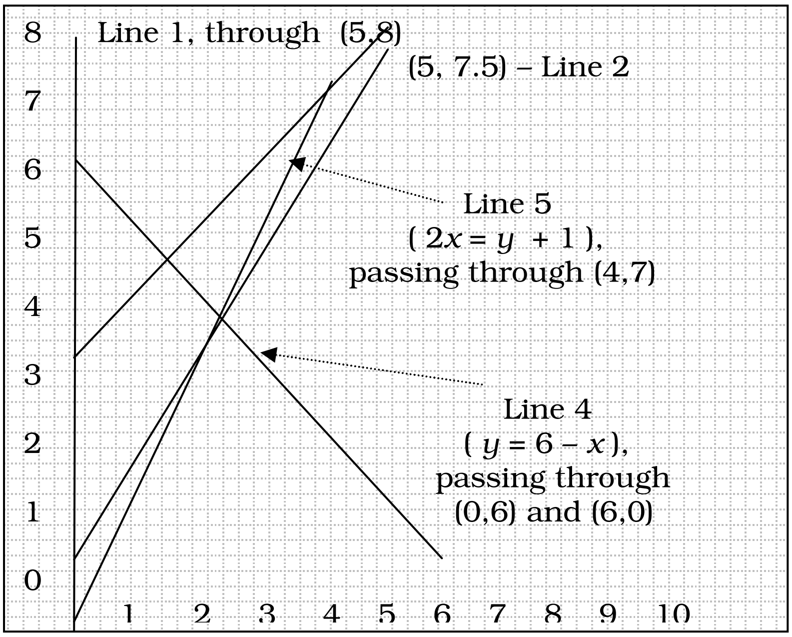
Work out whether each of the following lines has a positive, or a negative gradient. You are welcome to plot them, for practice; but try working each one out in your head (from the co-ordinate figures) if you can, and then see by plotting afterwards whether you were right.

1. (5,5) to (2,7)
2. (8,8) to (7,1)
3. (3,3) to (11,5)
4. (8,1) to (6,9)
5. (3,2) to (9,9)
6. (1,2) to (–3,1)
7. (–4,4) to (–2,2)
8. (8,0) to (6,–3)
9. (–7,1) to (–4,–2)
10. (–2,–4) to (–6,–3)

Suggested Answers to Activities

Activity One

1. $y = x + 3$
2. $y = 1.5\,x$
3. $y = x^2 - 1$
4. $y = 6 - x$
5. $2x = y + 1$



Notes :

1. A simple enough line : just add 3 to the *x*-value and keep going onwards & upwards in units of 1, at an angle of 45°

x	1	2	3
y (= x + 3)	4	5	6

2.

x	1	2	3
y (= 1.5 x)	1.5	3	4.5

3. This has an x^2 term in it, so please leave it alone for now

4. This has a distinctly different, though not particularly tricky feel to it, because the further you go along the *x*-values, the more you are taking away from *y*:

x	1	2	3
y (= 6 – x)	5	4	3

If you feel yourself 'losing your grip on reality' with it:

(a) Check the values, purely as an arithmetical exercise. If x is 3, y must also be 3 because 6 – 3 = 3.

(b) Bear in mind this is a perfectly realistic example. Imagine you receive £600 as a gift or win, and somehow you spend £100 each month: after 1 month you will have only £500 left, *etc*, until you run out after 6 months. A similar downward slope would apply to any other situation where 'the longer time goes on, the less is left', such as a vehicle getting through fuel at a steady rate during a long journey: 'the more of one, the fewer of the other'.

5. Faced with the equation $2x = y + 1$, we need to do a little rearranging before we can plan and plot our graph:

$$2x = y + 1$$
$$2x - 1 = y + 1 - 1 = y$$
$$y = 2x - 1 \qquad \text{('flip')}$$

Now to our table:

x	1	2	3
$2x$	2	4	6
$2x - 1$ ($= y$)	1	3	5
Co-ordinates	(1,1)	(2,3)	(3,5)

Activity Two

1. (b) 5, –3

2. (b) 1.5, 5

3. (a)

x	4	5	6	7	8	9
$y = x^2 - 11x + 26.25$	−1.75	−3.75	−3.75	−1.75	2.25	8.25

(c) 3.5, 7.5

Activity Three

1. (a) 15
(b) 10
(c) 5
(d) 13
(e) 5

2. (a) 10.8
   (b) 8.49
   (c) 12.2
   (d) 3.16

3. (a) (7, 9)
   (b) (1, –1)
   (c) (–3, –3)
   (d) (5.1, 1)
   (e) (2¼, 2¾)

Activity Three

1 (a) CD, GH, IJ

  (b) AB, EF

2 (a) 5

  (b) ¼

  (c) 3

  (d) ½

  (e) –2

  (f) –4

3.

	Gradient	Intercept on *y* axis
(a)	4	(0, –2)
(b)	1/3	(0, 7)
(c)	1	(0, –4/5)
(d)	–1	(0. 0.6)

4. AB: $y = 4x - 2$
   CD: $y = -2x + 3$ or $y = 3 - 2x$

5. AB: $y = -x + 2$ or $y = 2 - x$
   CD: $y = 3x$

6. AB: $y = \frac{1}{2}x - 1$ or $y = \frac{x}{2} - 1$

   CD: $y = \frac{5}{2}x - 5$ or $y = 2.5x - 5$

Activity Four

1.	Negative	6.	Positive
2.	Positive	7.	Negative
3.	Positive	8.	Positive
4.	Negative	9.	Negative
5.	Positive	10.	Negative

It's worth remembering that whichever order the two points are presented in, you can work these gradients out by identifying which point has the lower-valued *x*-coordinate, and treating that as coming 'first'. Having shuffled them if necessary, look to see whether the *y*-coordinate for the point further to the right lies further upward than the leftmost one.

Don't forget that an apparently 'bigger number' in an *x*-value may still mean that point is further to the left … if it's preceded by a minus sign.

In the case of Q.2, for example, (8,8) is further along from the origin than (7,1) is; so, reading the line from right to left, we turn the coordinate pairs round:

(7,1) to (8,8) (because $7 < 8$, on the *x*-axis)

We now find that $8 > 1$ on the *y*-axis, so this is a positive gradient; quite a steep one, as it happened.