Lesson Fifteen

Solving Equations

Aims

The aim of this lesson is to enable you to:

- solve linear equations
- solve linear equations from their graph
- solve simultaneous equations from their graphs
- solve simultaneous equations algebraically

Context

An equation combines two expressions and says they are equal to each other. The next lesson will show how algebra and equations can be put to work to solve problems in real-life situations.

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Equations

Everyday life is full of equations. If Tony and Cherie have a table that seats six, how many people can they invite to dinner (including themselves)?

Let the number of possible guests be x. Now we can say:

$$x + 2 = 6$$

If we replace x with the number 4, we have *solved* the equation. In our algebra so far, we have been working with expressions such as $3x - 5$ or $\left(\frac{x}{2} + by\right)$

We have just seen an example of an equation,

$$x + 2 = 6.$$

How does this differ from an expression?

The key thing to look out for is the equals sign in the middle. Hence the term 'equation'. An expression might be much longer (e.g. '$(3x + 5) \times (4x + 2) \times (3y - 4)$'), but unless there is an equals sign somewhere in the middle, it cannot be an equation.

An equation combines two expressions and says they are equal to each other.

Equation or identity?

Both equations and identities have equals signs. An **equation** is "true" for a certain set of values. In the example above

$$x + 2 = 6$$

is an equation. It only holds when $x=4$.

In the previous lesson we met the **identity**

$$(a + b)^2 = a^2 + 2ab + b^2$$

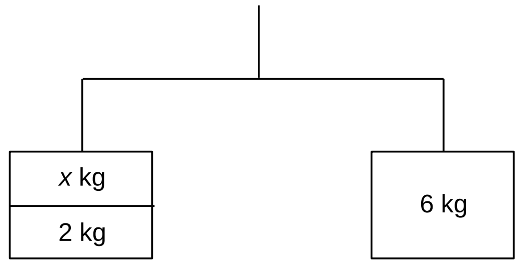
This is *always* true, for every value of a and b.

Solving Equations

Often when we look at algebraic equations, such as $x + 2 = 6$, we wish to replace the letter (or letters) with a number that solves the equation. In this example, if we replaced the letter 'x' with the number '4', we would have *solved* the equation.

Most equations are harder to solve, but there are certain basic principles that always apply. The equation is like a balance and the '=' sign can be considered the balancing point in the middle. Here is our simple equation as a balance:

$$x + 2 \text{ kg} \qquad = \qquad 6 \text{ kg}$$



You should be able to see at a glance that x kg must be 4 kg if the balance is going to be maintained. Whenever you see an algebraic equation, you assume that the two sides *do* balance and you look for numbers to replace the letters so that they will do so.

It is obviously useful if we can find actual values for letters in algebraic expressions, particularly when these occur in real-life situations. Solving an equation gets rid of the unknown elements. Don't forget: that to solve the equation, you must do the same thing to each side.

Making the Equation "True"

When you are asked to *solve* an equation, you should take it that a letter represents a specific unknown. 'Solving an equation' means finding the specific value of the letter which makes the equation true. The general idea is to change the equation in certain ways so that the letter representing the unknown is finally isolated.

There are (at least) two ways of thinking about 'changing' an equation. There is an 'official' way that involves 'doing the same to both sides'. How do we know what operations to do to both sides? The choice of operation depends upon the idea of 'inverse' operations. This just means 'opposite'. Addition and subtraction are a pair of 'inverse' operations: they 'undo' each other. If we add seven and then subtract seven, we get back to where we started. Similarly, multiplication and division are a pair of 'inverse' operations. If we multiply by ten and then divide by ten, we get back to where we started.

The other way of changing an equation also relies on this same idea of 'inverse' operations. The two most important techniques to remember are that when moving a term from one side of an equation to another:

1. plus becomes minus (and *vice versa*)
2. multiplication becomes division (and *vice versa*)

First of all, here are some illustrations of these techniques with ordinary arithmetic.

Example 1

Consider the (correct) sum: 2 + 3 = 5.

Move the three from the left to the right-hand side, and change the plus to a minus: 2 = 5 – 3. This **still** represents a correct sum. We could reverse the process to show that when the three moves back to the left-hand side, the minus becomes a plus.

Also consider the (correct) sum $2 \times 3 = 6$.

Move the three from the left to the right-hand side, and change the multiplication to division: $2 = \frac{6}{3} = 6 \div 3$. This **still** represents a correct sum. Reversing the process, if the three were to move back to the left-hand side, the division would become multiplication.

Example 2

Solve the following equations:

(a) $x + 3 = 8$ **(b)** $x - 2 = 4$

(c) $3x = 15$ **(d)** $\frac{x}{7} = 6$

	'Official' method	
(a)	$x + 3 = 8$ $x + 3 - 3 = 8 - 3$ $x + 0 = 5$ $x = 5$	Subtract 3 from both sides: But 3 -3 = 0. Adding zero has no effect: x is now isolated.
(b)	$x - 2 = 4$ $x - 2 + 2 = 4 + 2$ $x - 0 = 6$ $x = 6$	Add 2 to both sides: But -2 + 2 = 0. Adding or subtracting zero has no effect: x is now isolated.
(c)	$3x = 15$ $\frac{3x}{3} = \frac{15}{3}$ $\frac{x}{1} = 5$	Divide both sides by 3: The threes on the left-hand side cancel: Dividing by one has no effect: x is now isolated.

	$x = 5$	
(d)	$\frac{x}{7} = 6$ $\frac{x}{7} \times 7 = 6 \times 7$ $\frac{x}{1} = 42$ $x = 42$	Multiply both sides by seven: The sevens on the left-hand side cancel: Dividing by one has no effect: x is now isolated.

	Alternative method	
(a)	$x + 3 = 8$ $x = 8 - 3$ $x = 5$	Move the "3" to the other side, changing the plus to a minus. x is now isolated.
(b)	$x - 2 = 4$ $x = 4 + 2$ $x = 6$	Move the two to the other side, changing the minus to a plus: x is now isolated.
(c)	$3x = 15$ $x = \frac{15}{3} = 5$	Move the 3 to the other side, changing the implied multiplication to a division: x is now isolated.
(d)	$\frac{x}{7} = 6$ $x = 6 \times 7 = 42$	Move the seven to the other side, changing the division to a multiplication: x is now isolated.

Double-checking a solution is not only important, but quick and easy.

(a) Substitute $x = 5$ in the left-hand side of the original equation and see whether we get the right-hand side: $5 + 3 = 8$, so yes.

(b) Substitute $x = 6$ in the left-hand side of the original equation and see whether we get the right-hand side: $6 - 2 = 4$: yes.

(c) Substitute $x = 5$ in the left-hand side of the original equation and see whether we get the right-hand side: $3 \times 5 = 15$: yes.

(d) Substitute $x = 42$ in the left-hand side of the original equation and see whether we get the right-hand side: $\frac{42}{7} = 6$: yes.

It does not particularly matter whether you use the 'official' method or the alternative method. All that matters is that you learn to solve equations accurately, fluently and confidently.

The equations in Example 1 were simple in the sense that only one operation was involved in each equation. The following Example shows how to deal with two operations.

Example 3

Solve the following equations:

(a) $2x + 3 = 15$ **(b)** $4x - 1 = 19$

(c) $\frac{x}{3} + 1 = 5$ **(d)** $\frac{x}{5} - 2 = 5$

(a)	$2x + 3 = 15$ $2x = 15 - 3$ $2x = 12$ $x = \frac{12}{2} = 6$	Move the three to the other side, changing the plus to a minus: Move the two to the other side, changing the implied multiplication to a division: x is now isolated.
(b)	$4x - 1 = 19$ $4x = 19 + 1$ $4x = 20$ $x = \frac{20}{4} = 5$	Move the one to the other side, changing the minus to a plus: Move the four to the other side, changing the implied multiplication to a division: x is now isolated.
(c)	$\frac{x}{3} + 1 = 5$ $\frac{x}{3} = 5 - 1$ $\frac{x}{3} = 4$ $x = 4 \times 3 = 12$	Move the one to the other side, changing the plus to a minus: Move the three to the other side, changing the division to a multiplication: x is now isolated.
(d)	$\frac{x}{5} - 2 = 5$ $\frac{x}{5} = 5 + 2$ $\frac{x}{5} = 7$ $x = 7 \times 5 = 35$	Move the two to the other side, changing the minus to a plus: Move the five to the other side, changing the division to a multiplication: x is now isolated

Do not forget the double-checks!

(a) $2 \times 6 + 3 = 15$: correct.

(b) $4 \times 5 - 1 = 19$: correct.

(c) $\frac{12}{3} + 1 = 5$: correct.

(d) $\frac{35}{5} - 2 = 5$: correct.

One important point has just been glossed over. There were two operations in each of the above equations. How do we know which operation to 'undo' first? It turns out that when solving equations, we do BODMAS in reverse!

Thus in part (a), there is a multiplication and an addition on the left-hand side. BODMAS says that multiplication has priority over addition when evaluating the left-hand side. However, when solving the equation, we deal with the addition first and the multiplication second. If you look back at the other three parts, you will see that addition and subtraction are dealt with **before** multiplication and division: i.e. BODMAS in reverse.

The following Example further illustrates this principle of 'priority'.

Example 4

Solve the following equations:

(a) $\frac{x+1}{3} = 4$ **(b)** $2(x-3) = 10$ **(c)** $\frac{3x+2}{4} = 5$

(a) If we were to evaluate the left-hand side for a certain value of x, we would first of all note that there is an implied bracket: $\frac{(x+1)}{3}$. BODMAS would tell us to add one first (because of the bracket), and then divide by three. So when we are solving the equation, we reverse the operations by dealing with the three first, followed by the one.

$\frac{(x+1)}{3} = 4$ $x+1 = 4\times3$ $x+1 = 12$ $x = 12 - 1$ $x = 11$	Move the three to the other side, changing the division to multiplication: Move the one to the other side, changing the addition to subtraction: x is now isolated.

(b) Firstly, note that there is an implied multiplication sign between the two and the bracket. If we were to evaluate the left-hand side for a certain value of x, we would subtract three first (because of the bracket) and then multiply by two. To solve the equation, reverse the operations: deal with the two first and the three second.

$2(x-3) = 10$ $x - 3 = \frac{10}{2}$ $x - 3 = 5$ $x = 5 + 3$	Move the two to the other side, changing the multiplication to division: Move the three to the other side, changing the minus to plus: x is now isolated.

(c) First, insert the brackets implied by the fraction: $\frac{(3x+2)}{4} = 5$. Now, of we were to evaluate the left-hand side, we would start inside the bracket by multiplying by three and adding two, and then dividing by four. To solve the equation, reverse the operations: deal with the four first, then the two, and finally the three.

$\frac{(3x+2)}{4} = 5$	Move the four to the other side, changing the division to multiplication:
$3x + 2 = 5 \times 4$ $3x + 2 = 20$	Move the two to the other side, changing the plus to a minus:
$3x = 20 - 2$ $3x = 18$	Move the three to the other side, changing the (implied) multiplication to division:
$x = \frac{18}{3}$ $x = 6$	x is now isolated.

When there is an 'Unknown' on both Sides of the Equation

There are occasions when the 'unknown' letter appears in more than one place, including on both sides of the equation. The aim is still to isolate the unknown, but a little extra work is required to achieve this.

Example 5

Solve the following equations:

(a) $5x + 1 = 3x + 9$ **(b) $2(x - 3) = 5(x - 6)$**

(c) $3(x + 2) + 2(x - 1) = 6$

(a) The extra work comes at the beginning: we need to 'collect like terms' first, in a special way. Put the terms in x on one side and the remaining terms on the other side. The choice of side does not affect the answer, but it is much more convenient to avoid unnecessary minus signs by choosing to put the x terms on the left-hand side in this case.

$5x + 1 = 3x + 9$	Move the $3x$ to the left-hand side. Change the (implied) positive to a negative. Also, move the one to the right-hand side, changing the plus to a minus:
$5x - 3x = 9 - 1$	Simplify both sides:
$2x = 8$	Move the two to the other side, changing the multiplication to division:
$x = \frac{8}{2}$ $x = 4$	x is now isolated.

Double-check: when $x = 4$, the left-hand side of the original equation is $5 \times 4 + 1 = 21$. The left-hand side is $3 \times 4 + 9 = 21$. This confirms the solution.

(b) The first step is to multiply out the brackets. Then we need to collect terms. In this case is it more convenient to put the x terms on the **right**-hand side.

$2(x - 3) = 5(x - 6)$	Multiply out the brackets:
$2x - 6 = 5x - 30$	Move the $2x$ to the right-hand side, changing the (implied) positive to a negative, move the 30 to the left-hand side, changing the minus to a plus:
$-6 + 30 = 5x - 2x$	Simplify both sides:
$24 = 3x$	Move the three to the left-hand side, changing the multiplication to division:
$\frac{24}{3} = x$	
$x = 8$	x is now isolated.

Double-check: when $x = 8$, both sides of the original equation become 10.

(c) The initial extra work is again multiplying out the brackets.

$3(x + 2) + 2(x - 1) = 6$	Multiply out the brackets:
$3x + 6 + 2x - 2 = 6$	Collect like terms on the left-hand side:
$5x + 4 = 6$	Move the four to the other side, changing the plus to a minus:
$5x = 6 - 4$	Move the five to the other side, changing the multiplication to division:
$5x = 2$	x is now isolated.
$x = \frac{2}{5}$ or 0.4	

Activity 1

Solve the following equations:

1. (a) $x + 7 = 12$ (b) $x - 3 = 12$
   (c) $4x = 12$ (d) $\frac{x}{2} = 12$
2. (a) $3x + 4 = 10$ (b) $6x - 2 = 10$
   (c) $\frac{x}{7} + 3 = 10$ (d) $\frac{x}{5} - 4 = 10$
3. (a) $3(x + 1) = 6$ (b) $\frac{x-2}{5} = 6$
   (c) $2(7x - 11) = 6$ (d) $\frac{2(2x+3)}{5} = 6$
4. (a) $10x + 3 = 3x + 45$ (b) $5x - 2 = 2x + 10$
   (c) $4x + 5 = 8x - 19$ (d) $2x - 7 = 9x - 63$
5. (a) $2(x + 1) + 3(x + 5) = 37$ (b) $5(x - 2) + 4(x + 3) = 83$
   (c) $3(2x + 1) + 2(3x - 1) = 25$ (d) $4(3x - 2) - 2(3x - 1) = 12$

Solving Equations Using Graphs

You may have noticed that all the expressions we have dealt with in this lesson have been linear, that is you could rearrange them and write them in the form $mx + c$ where m and c are numerical constants. We know from Lesson Nine that the graph of $y = mx + c$ is a straight line. We are now going to see what "solve an equation" means graphically.

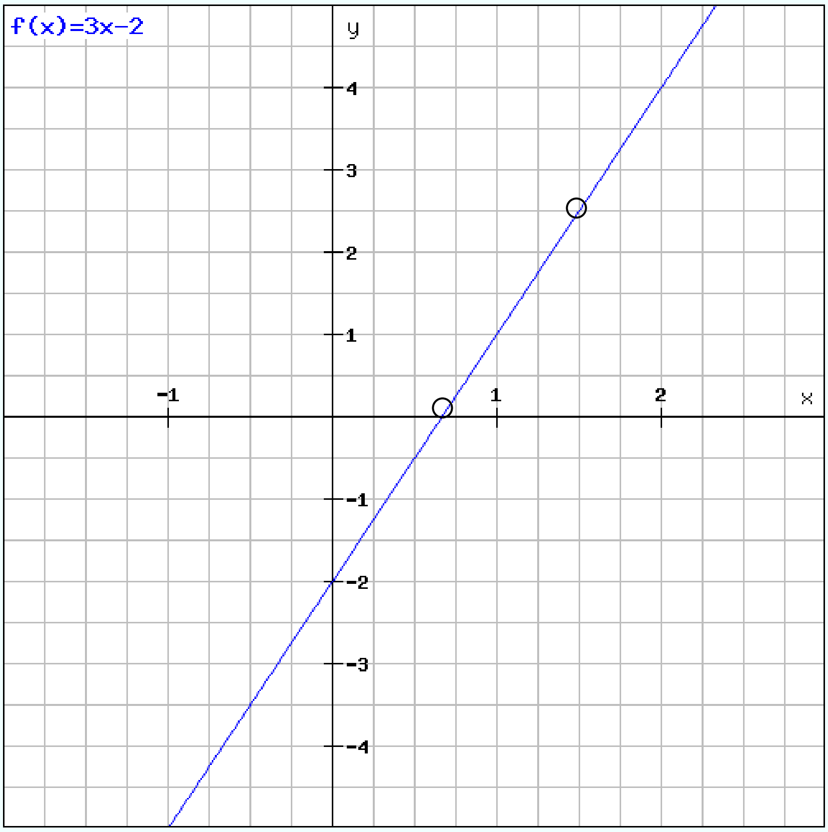
Example

Plot the graph of $y = 3x - 2$ for values of x from –1 to 2. Use your graph to write down the solution of $3x - 2 = 0$.

Construct a table of values as we have done before. As we know the graph is a straight line, we do need many points (2 would do but we include 2 extras to ensure we have not made a mistake).

x	-1	0	1	2
$y = 3x - 2$	$3 \times -1 - 2 = -5$	-2	1	4

And plot the points on a graph.

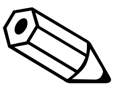


The graph shows all the points $3x - 2 = y$. We want to know the solution of $3x - 2 = 0$. In other words, we need the x value when $y = 0$. We can read this off the graph: $x \approx 0.6$ ("≈" mean "approximately equals") when $y = 0$.

We could solve other equations from this graph. For example to find the solution of $3x - 2 = 2.5$ we look for the x value that gives us a y value of 2.5.

Activity 2

Plot the graph of $y = 5 - 2x$ for values of x from –1 to 4 and use It to solve the following equations:



1. $5 - 2x = 0$
2. $5 - 2x = 3.5$
3. $5 - 2x = -2$

Solving Simultaneous Equations

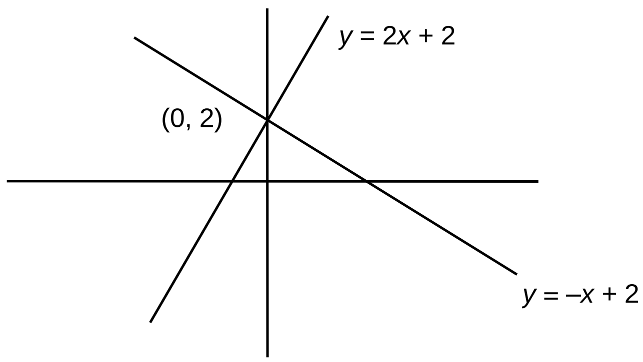
Using graphs also gives allows us to solve sets of equations, by finding the points of intersection of their graphs.

Example

Solve the simultaneous equations $y = 2x + 2$ and $y = -x + 2$.

"Simultaneous" means "at the same time". So we are looking for the values of x and y for which $y = 2x + 2$ and $y = -x + 2$ at the same time.

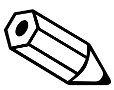
We can plot the graphs of $y = 2x + 2$ and $y = -x + 2$ as we did in Lesson Nine.



The graph shows us that the simultaneous equations are both satisfied at the point (0, 2).

Activity 3

Use a graphical method to solve these simultaneous equations. (Draw both axes from –4 to 16.)



1. $y = x + 1$ $\quad\quad y = 3x - 4$
2. $2y = 12 - x$ $\quad\quad 3y = x - 2$ hint: rearrange these first to get $y = 6 - \frac{1}{2}x$ for the first equation and $y = \ldots$
3. $2y = 3 - x$ $\quad\quad 2y = 4x - 7$

Solving Simultaneous Equations Algebraically

Consider the following problem. You tell your greengrocer that you have £1 to spend on apples and bananas. He says you could buy four apples and three bananas or you could buy six apples and two bananas. In either case, you would get no change from £1, i.e. the bill is 100p exactly.

Activity 4	Is it possible to tell from this what the price of an apple is? What about the price of a banana?

Because it is a simple practical problem, you might be able to work that one out in your head, but the important thing is to establish a method for sorting out more difficult problems. This is where algebra comes in handy. It is a 'shorthand' method for expressing problems of this kind.

To start with, the price of apples is unknown, so we represent it by a letter – perhaps the letter 'x'. The price of bananas is also unknown and is probably different from the price of apples, so we give it a different letter, perhaps 'y'. Now we write down what we know so far:

$$4x + 3y = 100 \text{ (pence)}$$

$$6x + 2y = 100 \text{ (pence)}$$

These are **simultaneous equations**. You know what an equation is already. Equations are *simultaneous* when they contain the same set of unknowns (here, x and y) and show *different* ways of relating the unknowns to each other.

This is usually a great help. You should assume that it is going to be possible to relate those two pieces of information to each other in such a way that we can get rid of the unknowns. In other words, it should be possible to work out the price of apples and bananas. Incidentally, apples cost 10p each and bananas 20p, but don't worry about that now.

In the above example, you will see that both combinations add up to exactly 100 pence. In other words, we know that $4x + 3y = 6x + 2y$. Is this vital? Would we have had simultaneous equations if $4x + 3y$ had been only 90 pence?

Yes, we would. To make the two sides balance, we might simply have added the extra figure:

$$4x + 3y + 10 = 6x + 2y \ (=100).$$

The vital thing is that *both* unknowns (and no others) appear on *both* sides. Remember that if you only have *one* unknown in an equation, you can find at least one solution straightaway (e.g. if $2x + 3 = 11$, then we know that $x = 4$).

Pairs of Simultaneous Equations

‘Simultaneous equations’ have more than one ‘unknown’. In this course, you need to know how to deal with a pair of ‘simultaneous equations’ which have two unknowns.

Imagine that you are queuing in a café. You cannot see the Menu or Price List. The person two places in front of you orders two teas and three coffees and is charged £2.60. The person immediately in front orders two teas and five coffees and is charged £3.80. How much does one tea cost? How much does one coffee cost?

Change the pounds to pence for convenience. Let the cost of one tea be t pence, and cost of one coffee be c pence. The information can be translated into algebra as:

Order 1: $2t + 3c = 260$
Order 2: $2t + 5c = 380$

Some natural questions arise.

1. Which Order costs more? Answer: the second.
2. How much more? Answer: 120 pence.
3. Why does the second Order cost more? It can be nothing to do with the teas: there are the same number of teas in both Orders. Answer: the second Order costs more because there are extra coffees in the second Order.
4. How many extra coffees? Answer: there are two extra coffees. And the bill is £1.20 more.

We are now in a position to solve the problem. Two coffees must cost 120 pence, so one coffee must cost 60 pence.

Believe it or not, that is the essence of simultaneous equations. We still need to find the cost of coffee. This requires some work, but all of the techniques should be familiar to you from the previous lesson: Solving (ordinary) Equations. The details are omitted for the present, so that the main strategy remains as clear as possible.

The above example was especially simple. What made it so was the fact that the number of teas was the same in both orders. However, even if the number of teas (and coffees) is different in both orders, we can always make them the same. This is the essence of the strategy for solving Simultaneous Equations.

Example 1

Find the value of *c* if $t + 3c = 22$ **and** $2t + 5c = 39$**.**

There are always two ways of proceeding. In this case, one way is much easier than the other. Translate the first equation back into teas and coffees. Note that there is an implied "1" in front of the *t* in the first equation: one tea and three coffees costs 22. If we were to double this Order, it would become:

$$2t + 6c = 44$$

Notice that everything has been doubled: one tea has become two teas, three coffees have become six coffees, and 22 has become 44. Why did we do this? To make the number of teas the same in both Orders:

$$2t + 6c = 44$$
$$2t + 5c = 39$$

Now the situation is the same as it was in the first 'simple' example. The first order is five more than the second because there is one more coffee. So *c* must be 5.

Example 2

Find the value of *t* given that $3t + 5c = 21$ **and** $4t + 7c = 29$

To find *t*, we need to make the numbers in front of *c* the same. This is harder than Example 1, where we only needed to change one equation. In this case, we need to change both equations. The strategy is to multiply the first 'order' by seven and the second 'order' by five.

1. Multiply **everything** in the first equation by 7: $21t + 35c = 147$
2. Multiply **everything** in the second equation by 5: $20t + 35c = 145$

Now the top equation is 2 more than the bottom because there is one extra t. Therefore t must be 2.

There is something familiar happening here. The numbers in front of c were 5 and 7, and we ended up with the number 35. This is exactly what would have happened if we had been adding (or subtracting) fractions with denominators of 5 and 7, e.g.

$$\frac{1}{5} + \frac{1}{7} = \frac{1 \times 7}{5 \times 7} + \frac{1 \times 5}{7 \times 5} = \frac{7}{35} + \frac{5}{35} = \frac{7+5}{35} = \frac{12}{35}$$

Recall that 35 is the Lowest Common Denominator of the two original fractions, in other words the Lowest Common Multiple of 5 and 7. In this case, we have obtained the Lowest Common Multiple by multiplying together the two numbers 5 and 7. However, this is not always the case. If we were to add, say, $\frac{1}{4} + \frac{1}{6}$, we would change the denominator of both fractions to 12, which is the Lowest Common Multiple of 4 and 6.

Example 3

Find the value of c given that $4t + 11c = 19$ **and** $6t + 17c = 29$

The aim is to make the numbers in front of the t the same in both cases. This number will actually be the Lowest Common Multiple of 4 and 6, i.e. 12.

1. Multiply **everything** in the first equation by 3: $12t + 33c = 57$
2. Multiply **everything** in the second equation by 2: $12t + 34c = 58$

The bottom equation is now one more than the top equation because there is one extra c. Therefore $c = 1$.

To find the Value of Both Unknowns

We need to fill in the rest of the details, most of which are revision. You need to know how to find the value of both the unknowns. It is also helpful to perform a quick double-check. It is common to use the letters x and y for the 'unknowns' in simultaneous equations.

Example 4

Solve the simultaneous equations $15x+12y=90$ **and** $7x+8y=54$

We have two choices: either make the numbers in front of x the same, or make the numbers in front of the y the same. It makes no difference to the end result. In this Example, the choice is arbitrary. So choose to make the numbers in front of y the same. The Lowest Common Multiple of 8 and 12 is 24. So we need to make the numbers in front of each y into 24.

1. Multiply **everything** in the first equation by 2: $30x+24y=180$
2. Multiply **everything** in the second equation by 3: $21x+24y=162$

The top equation is 18 more than the bottom equation because there are nine extra xs.
If $9x=18$ then $x = 2$.

Now we choose one of the original equations, say the first one. Substitute $x = 2$:

$15\times 2+12y=90$

So $30+12y=90$

$12y=90-30$

$12y=60$

$y=\frac{60}{12}$

$y=5$

Now double-check. Use the second equation (the one we did **not** use to find y).

Substitute $x = 2$ and $y = 5$ in the left-hand side to give $7\times 2+8\times 5=14+40=54$, so the result is the original right-hand side, which confirms that the answers are correct.

In all cases so far, we have eventually **subtracted** two equations. For instance, in Example 4:

$$
\begin{array}{rl}
 & 30x + 24y = 180 \\
- & 21x + 24y = 162 \\
\hline
 & 9x \qquad = 18
\end{array}
$$

Working has taken place in three columns. In the left-hand column, $30x - 21x = 9x$. In the middle column, $24y - 24y = 0y = 0$, so we do not need to write anything in the middle column of the answer. In the right-hand column, $180 - 162 = 18$.

Positive and Negative Numbers

So far, all the numbers in the simultaneous equations have been positive. This is not always the case. Care must be taken when both positive and negative numbers are present.

Example 5

Solve the simultaneous equations: $19x - 3y = 140$**,** $17x - 3y = 124$

This is a 'simple' example since the number in front of the *y* is the same. So we subtract the two equations as before, working in three columns:

$$
\begin{array}{rl}
 & 19x - 3y = 140 \\
- & 17x - 3y = 124 \\
\hline
 & 2x \qquad = 16
\end{array}
$$

Look carefully at the middle column. The working is $-3y - (-3y)$. The two minuses make a plus, so that the result is $-3y + 3y = 0y = 0$.

So *x* must be 8. Substitute in the first equation: $19 \times 8 - 3y = 140$, so $152 - 3y = 140$. Move the 3*y* to the right-hand side and the 140 to the left-hand side to give $152 - 140 = 3y$. So $12 = 3y$, so $y = 4$. Check in the second equation: $17 \times 8 - 3 \times 4 = 124$, as required.

There are occasions when we **add** the two equations.

Example 6

Solve the simultaneous equations: $3x + 5y = 58$**,** $7x - 5y = 2$**.**

This is also a 'simple' example because the numbers in front of y are the same, although, strictly speaking, they are different because one is positive and the other is negative. If we line up the two equations, as before:

$$\begin{array}{rl} & 3x + 5y = 58 \\ + & 7x - 5y = 2 \\ & \text{---------------} \\ & 10x \qquad = 60 \end{array}$$

We now **add** the two equations, by working again in three columns. In the left-hand column, $3x + 7x = 10x$. In the middle column, $5y + (-5y) = 0y = 0$, so we do not need to write anything in the middle column of the answer. In the right-hand column, 58 + 2 = 60.

The rest of the solution would now be routine: $x = 60 \div 10 = 6$; substitute in the first equations: $18 + 5y = 58$, so $5y = 40$, y = 8. Then double-check using the second equation, $7 \times 6 - 5 \times 8 = 42 - 40 = 2$, as required.

When do we Add, when do we Subtract?

This question is linked with the fact that there are always two possible strategies for making the 'initial breakthrough': finding the value of one of the unknowns. The following two examples are intended to show **both** the available strategies.

Example 7

Solve the simultaneous equations: $2x - 3y = 8$**,** $3x - 5y = 11$

We have a choice, and it makes little difference. We can 'eliminate' y by making the numbers in front of y the same. The Lowest Common Multiple of 3 and 5 is 15. So multiply everything in the first equation by 5, and everything in the second equation by 3, to give the new equations, which need to be subtracted:

$$\begin{array}{rl} & 10x - 15y = 40 \\ - & 9x - 15y = 33 \\ & \text{------------------} \\ & x \qquad = 7 \end{array}$$

Note $10x - 9x = 1x = x$. Also, $15y-(-15y)=-15y+15y=0y=0$. If we substitute $x = 7$ in either of the equations, we find that $y = 2$.

We could have 'eliminated' x instead of y by making the numbers in front of x the same, which would be 6 (the Lowest Common Multiple of 2 and 3). We would need to multiply everything in the first equation by 3, and multiply everything in the second equation by 2. The two new equations would now be:

$$\begin{array}{rl} & 6x-9y=24 \\ - & 6x-10y=22 \\ \hline & y \;\; =2 \end{array}$$

The equations are again subtracted. Examine the working in the middle column very carefully: the two minuses make a plus: $-9y-(-10y)=-9y+10y=1y=y$. We would now substitute $y = 2$ in either of the equations and find that $x = 7$.

Notice that we have subtracted the equations in both alternative strategies.

Example 8

Solve the simultaneous equations: $2x+3y=13$, $3x-5y=10$.

We can 'eliminate' x by making the numbers in front of x the same: in fact 6: the Lowest Common Denominator of 2 and 3. Multiply everything in the first equation by 3 and multiply everything in the second equation by 2:

$$\begin{array}{rl} & 6x+9y=39 \\ - & 6x-10y=20 \\ \hline & 19y \;=19 \end{array}$$

Look carefully at the working in the middle column: $9y-(-10y)=9y+10y=19y$.
So y must be 1, and we would substitute this in either equation to find that x is 5.

However, look what happens if we 'eliminate' y. The Lowest Common Multiple of 3 and 5 is 15. So multiply everything in the first equation by 5, and multiply everything in the second equation by 3:

$$
\begin{array}{r}
10x + 15y = 65 \\
+\ \ 9x - 15y = 30 \\
\hline
19x \qquad = 95
\end{array}
$$

Notice that in this case we **add** the two new equations. The aim is to eliminate *y*, and adding accomplishes this: $15y + (-15y) = 0y = 0$.

We would conclude that *x* must be 5, and then substitute in either of the equations to find out that *y* is 1.

Now compare the overall strategies for Examples 7 and 8. There are two available strategies for each Example. In Example 7, subtraction was required for both strategies, whereas in Example 8, one strategy required subtraction and the other required addition.

We return to the question: how do you know when to add and when to subtract? You can either use the rule:

- *subtract* when the numbers in front of one letter are identical, including the sign
- *add* when the numbers in front of one letter are the same but with a different sign

or the less detailed, and more practical, rule:

- do whatever you need to do to 'eliminate' one letter.

The next Example concentrates on just this overall strategy: we are not interested in the details of the final answer.

Example 9

Do not solve any of the following sets of simultaneous equations. Merely outline the possible strategies for solution.

(a) $7x + 3y = 10$, $21x + 5y = 26$
(b) $17x - 10y = 7$, $19x - 15y = 4$
(c) $11x + 18y = 29$, $2x - 3y = -1$
(d) $13x + 9y = 22$, $7x - 12y = -5$

	To 'eliminate' *x*	**LCM**	**To 'eliminate' *y***	**LCM**
(a)	Multiply first equation by 3. **Subtract**.	21	Multiply first equation by 5. Multiply second equation by 3. **Subtract.**	15
(b)	Multiply first equation by 19. Multiply second equation by 17. **Subtract**.	17×19	Multiply first equation by 3. Multiply second equation by 2. **Subtract.**	30
(c)	Multiply first equation by 2. Multiply second equation by 11. **Subtract.**	22	Multiply second equation by 6. **Add**.	18
(d)	Multiply first equation by 7. Multiply second equation by 13. **Subtract**.	7×13	Multiply first equation by 4. Multiply second equation by 3. **Add.**	36

The final example is to show that:

- the form of simultaneous equations can vary
- the answers are not always whole numbers.

Example 10

Solve the following simultaneous equations.

(a) $2x = y + 10$, $4x = 11 - y$
(b) $3p - 2q - 1 = 0$, $15p = 4 + 6q$

The 'standard format' consists of the two letters on the left-hand side and the 'constant' term (not attached to any letter) on the right-hand side. Moreover, the order of the letters on the left-hand side must be the same for both equations. These rules therefore mean moving terms from one side to another, and reversing the order of some terms.

(a) The 'standard format' for the equations is: $2x - y = 10$, $4x + y = 11$.

Notice that the numbers in front of the *y*'s are both one, although they are of opposite sign. If we add the equations, we will eliminate *y*:

$$\begin{array}{rl} & 2x - y = 10 \\ + & 4x + y = 11 \\ \hline & 6x \quad\;\; = 21 \end{array}$$

So $x = \frac{21}{6}$. But we should always try to 'cancel down' final answers that are a fraction. Divide top and bottom by three to give $x = \frac{7}{2}$. It is perfectly acceptable to write this as a mixed number: $x = 3\frac{1}{2}$, or even (in this particular case) as a decimal: $x = 3.5$.

Substitute $x = \frac{7}{2}$ in the first equation: $2 \times \frac{7}{2} - y = 10$, so that $7 - y = 10$. Move the y to the right-hand side and the ten to the left-hand side, to give: $7 - 10 = y$. y must therefore be –3.

Double-check in the second equation: $4 \times \frac{7}{2} + (-3) = 14 - 3 = 11$, as required.

(b) Writing the equations in 'standard form': $3p - 2q = 1$, $15p - 6q = 4$. If we decide to 'eliminate' q, then all we need to do is to multiply the first equation by three. We then subtract, since the numbers in front of the y's are then exactly the same, including the sign. Notice that it is more convenient to subtract the (new) first equation from the second:

$$\begin{array}{rl} & 15p - 6q = 4 \\ - & \ \ 9p - 6q = 3 \\ \hline & \ \ 6p \qquad = 1 \end{array}$$

It is better not to use a calculator, even if one is available. It is much better to write $p = \frac{1}{6}$. This is simple and exact. If we were to do the division $1 \div 6$ on a calculator:

- the answer would no longer be exact
- we would need to make decisions about decimal places or significant figures, or else use the notation for recurring decimals.

Substitute $p = \frac{1}{6}$ in the first equation: $3 \times \frac{1}{6} - 2q = 1$

But $\frac{3}{6} = \frac{1}{2}$, so that $\frac{1}{2} - 2q = 1$.

Move the $2q$ to the right-hand side and the 1 to the left-hand side: $\frac{1}{2}-1=2q$

Now $\frac{1}{2}-1=-\frac{1}{2}$, so that $-\frac{1}{2}=2q$.

Divide both sides by 2: $q=-\frac{1}{2}\div 2=-\frac{1}{2}\times\frac{1}{2}=-\frac{1}{4}$

Double-check in the second equation:

$15\times\frac{1}{6}-6\times\left(-\frac{1}{4}\right)=\frac{5}{2}+\frac{3}{2}=\frac{8}{2}=4$, as required.

Activity 5

In questions 1 to 4, solve the following simultaneous equations and check your answers.

1 (a) $3x+7y=33$, $9x+7y=113$
(b) $5x+3y=69$, $5x+9y=87$
(c) $7x+5y=54$, $2x-5y=9$
(d) $18x-4y=46$, $13x-4y=31$

2 (a) $3x+7y=46$, $x+9y=42$
(b) $5x+18y=51$, $x+2y=7$
(c) $11x+8y=62$, $7x+4y=34$
(d) $10x-7y=83$, $2x-13y=5$
(e) $8x-3y=47$, $11x-12y=41$
(f) $5x-3y=2$, $17x+6y=104$

3 (a) $5x+3y=52$, $7x+11y=100$
(b) $11x+8y=62$, $23x+12y=106$
(c) $2x-5y=13$, $7x-y=62$
(d) $15x-19y=22$, $20x-23y=34$
(e) $4x+7y=49$, $11x-5y=62$
(f) $7x+6y=72$, $9x-8y=14$

4 (a) $3h-4k=3$, $2h-3k=1$
(b) $6p+7q=90$, $p+5q=38$
(c) $3m+20n=70$, $7m-30n=10$
(d) $2a=5b+3$, $3a=7b+6$
(e) $10f=17+g$, $8f=6-3g$

If you feel you need extra practice at deciding on the **strategy** for solving simultaneous equations, try the following three questions.

5. Do not solve any of the following sets of simultaneous equations. None of the equations in this question requires multiplication. Just decide whether you would ADD or SUBTRACT each pair of equations.

   (a) $5x+7y=12$, $3x+7y=10$.
   (b) $18x+5y=33$, $18x-11y=7$
   (c) $4x-9y=3$, $13x-9y=30$
   (d) $33x+23y=56$, $35x-23y=13$

6. Each pair of equations in this question can be solved by: multiplying **one** of the equations by a certain number then adding **or** subtracting the pair of equations.

   Do not solve any of the following sets of simultaneous equations. Just outline the **strategy** for solving each pair. Each of your answers will therefore look something like:

   "Multiply the second equation by 3, and then subtract the two equations".

   (a) $2x-19y=81$, $x-44y=6$
   (b) $31x+4y=35$, $29x+16y=45$
   (c) $12x+17y=137$, $6x-37y=23$
   (d) $2x+13y=33$, $3x-26y=4$

7. Each pair of equations in this question can be solved by multiplying **both** of the equations by certain numbers then adding **or** subtracting the pair of equations.

   Do not solve any of the following sets of simultaneous equations. Just outline one **strategy** for solving each pair. (There are **two** possible strategies for each pair of equations: both are presented in the Answers.) Each of your answers will therefore look something like:

   "Multiply the first equation by 8, multiply the second equation by 3 and then add the two equations".

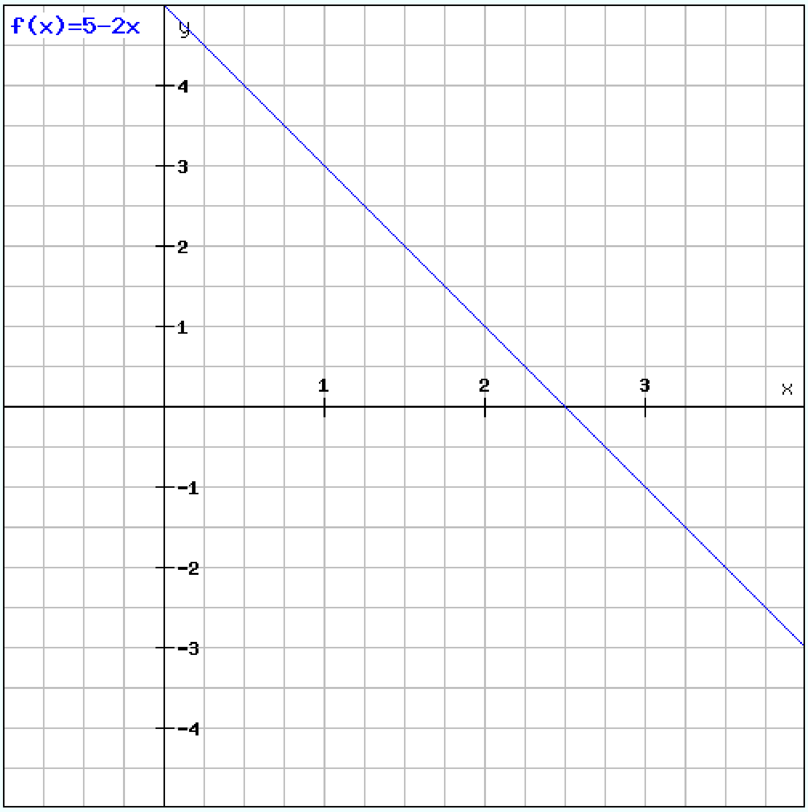
   (a) $5x+3y=8$, $8x+7y=15$
   (b) $7x+12y=26$, $5x-8y=2$
   (c) $12x+3y=15$, $18x-5y=13$
   (d) $5x-8y=7$, $3x-7y=2$

Suggested Answers to Activities

Activity One

1.	(a) $x = 5$	(b) $x = 15$	(c) $x = 3$	(d) $x = 24$
2.	(a) $x = 2$	(b) $x = 2$	(c) $x = 49$	(d) $x = 70$
3.	(a) $x = 1$	(b) $x = 32$	(c) $x = 2$	(d) $x = 6$
4.	(a) $x = 6$	(b) $x = 4$	(c) $x = 6$	(d) $x = 8$
5.	(a) $x = 4$	(b) $x = 9$	(c) $x = 2$	(d) $x = 3$

Activity Two



1. $x = 2.5$
2. $x = 0.75$
3. $x = 3.5$

Activity Three

1. $x = \frac{5}{2}, y = \frac{7}{2}$
2. $x = 8,\ y = 2$
3. $x = 2,\ y = 0.5$

Activity Four Yes; apples cost 10p each and bananas cost 20p each.

Activity Five

1. (a) $x = 13\frac{1}{3}$, $y = -1$ (b) $x = 12$, $y = 3$
   (c) $x = 7$, $y = 1$ (d) $x = 3$, $y = 2$

2. (a) $x = 6$, $y = 4$ (b) $x = 3$, $y = 2$
   (c) $x = 2$, $y = 5$ (d) $x = 9$, $y = 1$
   (e) $x = 7$, $y = 3$ (f) $x = 4$, $y = 6$

3. (a) $x = 8$, $y = 4$ (b) $x = 2$, $y = 5$
   (c) $x = 9$, $y = 1$ (d) $x = 4$, $y = 2$
   (e) $x = 7$, $y = 3$ (f) $x = 6$, $y = 5$

4. (a) $h = 5$, $k = 3$ (b) $p = 8$, $q = 6$
   (c) $m = 10$, $n = 2$ (d) $a = 9$, $b = 3$
   (e) $f = \frac{3}{2} = 1\frac{1}{2} = 1.5$, $g = -2$

5. (a) Subtract (b) Subtract
   (c) Subtract (d) Add

6. (a) Multiply the second equation by two, and then subtract the two equations.
   (b) Multiply the first equation by four, and then subtract the two equations.
   (c) Multiply the second equation by two, and then subtract the two equations.
   (d) Multiply the first equation by two, and then add the two equations.

7 (a) Multiply the first equation by 8, multiply the second equation by 5, and then subtract the two equations.

   OR:

   Multiply the first equation by 7, multiply the second equation by 3, and then subtract the two equations.

   (b) Multiply the first equation by 5, multiply the second equation by 7, and then subtract the two equations.

   OR:

Multiply the first equation by 2, multiply the second equation by 3, and then add the two equations.

(c) Multiply the first equation by 3, multiply the second equation by 2, and then subtract the two equations.

OR:

Multiply the first equation by 5, multiply the second equation by 3, and then add the two equations.

(d) Multiply the first equation by 3, multiply the second equation by 5, and then subtract the two equations.

OR:

Multiply the first equation by 7, multiply the second equation by 8, and then subtract the two equations.